Enumeration of class 2 associative algebras over finite fields

Von der
Carl-Friedrich-Gauß-Fakultät
der Technischen Universität Carolo-Wilhelmina zu Braunschweig

zur Erlangung des Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigte Dissertation

von
Morten Wesche
geboren am 31.08.1989
in Wolfenbüttel

Eingereicht am: 06.06.2018
Disputation am: 07.09.2018
1. Referentin: Prof. Dr. Bettina Eick
2. Referent: Prof. Dr. Michael Vaughan-Lee

2018

Contents

Chapter 1

Introduction

An ***associative $\mathbb{F}$-algebra*** $\mathcal{A}$ is an algebraic structure that is equipped with compatible operations of addition, associative multiplication, and a scalar multiplication by elements in the field $\mathbb{F}$. The multiplication and the addition give $\mathcal{A}$ the structure of a ring; The addition and the scalar multiplication give $\mathcal{A}$ the structure of a vector space over $\mathbb{F}$. Note that we do not require $\mathcal{A}$ to have an identity element.
We denote by $\mathcal{A}^l$ the ideal of $\mathcal{A}$ generated by all products of length l in $\mathcal{A}$. The ideal $\mathcal{A}^l$ is named l-th ***power ideal***. We call an algebra $\mathcal{A}$ ***nilpotent of class*** c if it has a power ideal series

$$\mathcal{A} = \mathcal{A}^1 \gneq \mathcal{A}^2 \gneq \ldots \gneq \mathcal{A}^c \gneq \mathcal{A}^{c+1} = \{0\} \tag{1.1}$$

terminating after $c+1$ steps in $\{0\}$. The dimension of $\mathcal{A}$ is the ***dimension*** of $\mathcal{A}$ as a vector space over $\mathbb{F}$ and it is denoted by $\dim(\mathcal{A})$. Its ***rank*** is the minimal number of generators as an algebra and it is denoted by $\mathrm{rk}(\mathcal{A})$.
There are many examples for associative $\mathbb{F}$-algebras. Every polynomial ring over $\mathbb{F}$ is an associative $\mathbb{F}$-algebra. The set of all square matrices over $\mathbb{F}$ forms an associative $\mathbb{F}$-algebra. The universal enveloping algebra of a Lie algebra is an associative $\mathbb{F}$-algebra and can be used to study the structure of the Lie algebra. Similarly, the structure of groups can be studied by considering their group-algebras which are associative $\mathbb{F}$-algebras. Further, associative $\mathbb{F}$-algebras are used in a variety of different topics such as representation theory or cohomology.
Every associative $\mathbb{F}$-algebra $\mathcal{A}$ can be embedded into an associative $\mathbb{F}$-algebra $\mathcal{B}$ with an identity element. Then there exists a nilpotent ideal $\mathcal{J}(\mathcal{B})$ called the Jacobson radical such that the quotient $\mathcal{B}/\mathcal{J}(\mathcal{B})$ is a semisimple algebra (compare [30, Section 4.1]). Wedderburn's structure theorem yields a classification of semisimple algebras [30, Section 3.5]. However, an analogue structure theorem for nilpotent associative $\mathbb{F}$-algebras does not exist.
An algorithm by Eick and O'Brien [11] can be used to count the number of isomorphism types of nilpotent associative $\mathbb{F}$-algebras of class 2 over a fixed finite field $\mathbb{F}_q$. A classification of nilpotent associative $\mathbb{F}$-algebras of dimension $d \leq 4$ was obtained by de Graaf [6]. Beyond that, classifications of associative $\mathbb{F}$-algebras are available in special cases only.

In this thesis we consider nilpotent associative $\mathbb{F}$-algebras of class 2 and we investigate the number of isomorphism types of such algebras. We only consider algebras defined over finite fields.

Definition 1.1 *The number of isomorphism types of nilpotent associative $\mathbb{F}_q$-algebras of class 2, dimension d and rank r defined over a finite field $\mathbb{F}_q$ with q elements is denoted by $N_{d,r}(q)$.*

The aims of our thesis are:

1. Investigate the properties of $N_{d,r}(q)$.
2. Develop an algorithm to determine the numbers $N_{d,r}(q)$ for given d and r as a function in q.
3. Compute explicit functions $N_{d,r}(q)$ for small d and r.

The methods by Eick and O'Brien can be used to determine $N_{d,r}(q)$ for fixed d, r, and q. In this thesis we describe methods to determine $N_{d,r}(q)$ for fixed d and r, but for arbitrary q.
Vaughan-Lee [35] and Witty [38] introduced algorithmic methods to determine the number of isomorphism types of p-groups of exponent-p class 2 and order p^n as a function in p for fixed n. Vaughan-Lee [37] also published methods to determine the number of isomorphism types of non-associative $\mathbb{F}_q$-algebras of fixed dimension d as a function in q. These functions are also PORC.
Our algorithm uses the same overall approach as the algorithms by Vaughan-Lee and Witty: We translate the problem to finding numbers of orbits where $\mathrm{GL}(n, \mathbb{F}_q)$ acts on suitable sets. However, the details of our methods differ from theirs. The main ideas of this thesis are published in [13].

1.1 Results

Our first aim is the investigation of $N_{d,r}(q)$. For this purpose we need the term *PORC function* which was originally introduced by Higman [18]. A detailed introduction into PORC functions is given in Section 2.1.

Definition 1.2: ***PORC function (Higman [18])***
Let S be an infinite subset of $\mathbb{N}$. A function $f : S \to \mathbb{Q}$ is a ***polynomial on residue classes*** *(PORC) if there exists a natural number $m \in \mathbb{N}$ and polynomials $g_0, \dots, g_{m-1} \in \mathbb{Q}[x]$ such that*

$$f(x) = g_a(x) \quad \text{for all } x \in S \text{ with } x \equiv a \bmod m. \tag{1.2}$$

A PORC function is ***homogeneous of degree*** *g if its associated polynomials $g_0, \dots, g_{m-1}$ all have the same degree g. The number m is called the* ***modulus*** *of f.*

Now, we can formulate the following theorem giving the general structure of $N_{d,r}(q)$. The proof of Theorem 1.3 will follow from Theorem 4.6 and Corollary 4.9 and 5.12.

Theorem 1.3 *Let q be a prime power and let $d, r \in \mathbb{N}$. Then $N_{d,r}(q)$ is PORC.*

The next theorem states further properties of the PORC functions $N_{d,r}(q)$. It will be proved in Section 4.2. A conclusion from Theorem 1.4 is that nilpotent associative algebras of class 2 and rank r have dimension d with $r + 1 \le d \le r^2$.

Theorem 1.4 *Let q be a prime power and let $r \in \mathbb{N}$. Then*

1. $N_{d,r}(q) = 0$ *if* $d \notin \{r+1, \ldots, r+r^2\}$,
2. $N_{r+r^2,r}(q) = 1$, *and*
3. $N_{r+k,r}(q) = N_{r+r^2-k,r}(q)$ *for all* $k \in \{1, \ldots, r^2-1\}$.

Our second aim is the development of an algorithm that computes the PORC functions $N_{d,r}(q)$ for given d and r. This is done in Chapter 4 and 5. A `GAP` [15] implementation of our algorithm is available in the *ClassTwoAlg* package [12]. It is based on `GAP` 4.8.8. Some functions of our algorithm are based on ideas and algorithms by Vaughan-Lee, see for example [34, 37]. A summary of our algorithm is given in Algorithm 1 on page 31.

The third aim of this thesis is the explicit computation of $N_{d,r}(q)$ for small r and d. Based on our algorithm [12] we determine the PORC functions $N_{d,r}(q)$ for $1 \leq r \leq 5$ and all d. The determination of the PORC functions $N_{d,r}(q)$ for $r=1$ yields $N_{2,1}(q) = 1$. For $r=2$ and $r=3$ we exhibit the results in Theorems 1.5 and 1.6 below and we include an independent proof of Theorem 1.5 in Section 6. Partial results for $r=4$ are given in Theorem 1.7. All larger results of rank $r=4$ and all results of rank $r=5$ are too large to be exhibited here. They can be found in Appendix A.
As expected, the runtime of our algorithm grows significantly with the rank r. Using a PC with a 3,40 GHz Intel processor and 32 GB RAM, we determined the PORC polynomials $N_{d,r}(q)$ for $r \in \{2,3,4\}$ in a few seconds, while for $r=5$ it took about 10 hours.
Now we exhibit the PORC functions $N_{d,r}(q)$ for small ranks and dimensions in the following theorems. Throughout this thesis we apply the following rules when writing PORC functions: Whenever we have a greatest common divisor of two numbers we omit the letters "$\gcd$" and use round brackets "$(,)$" instead. Therefore, for some integer $z \in \mathbb{Z}$ and some polynomial f we write $(f(q), z)$ to denote the greatest common divisor of z and the value of f at q. For grouping terms together we use square brackets "$[\,]$".

Theorem 1.5 *Let $q = p^s$ be an arbitrary prime power. Then*

$N_{3,2}(q) = q - (q,2) + 5.$
$N_{4,2}(q) = 3 \cdot q - (q,2) + 6.$

Theorem 1.6 *Let $q = p^s$ be an arbitrary prime power. Then*

$$N_{4,3}(q) = 2 \cdot q + [-2 \cdot (q,2) + 11]\,.$$

$$\begin{aligned} N_{5,3}(q) = q^6 &+ q^5 + 3 \cdot q^4 \\ &+ 6 \cdot q^3 + [-2 \cdot (q,2) + 18] \cdot q^2 + [-7 \cdot (q,2) + (q-1,3) + 38] \cdot q \\ &+ [-10 \cdot (q,2) - 1/2 \cdot (q,3) + 89/2]\,. \end{aligned}$$

$$\begin{aligned} N_{6,3}(q) = q^{10} &+ q^9 + 3 \cdot q^8 + 5 \cdot q^7 + 8 \cdot q^6 + 13 \cdot q^5 \\ &+ [-2 \cdot (q,2) + 29] \cdot q^4 + [-6 \cdot (q,2) + 48] \cdot q^3 \\ &+ [-15 \cdot (q,2) - 1/2 \cdot (q,3) + 2 \cdot (q-1,3) + (q-1,4) + 177/2] \cdot q^2 \\ &+ [-22 \cdot (q,2) - 1/2 \cdot (q,3) + 4 \cdot (q-1,3) + 2 \cdot (q-1,4) + 223/2] \cdot q \\ &+ [-20 \cdot (q,2) - 1/2 \cdot (q,3) + (q-1,3) + (q-1,4) + 173/2]\,. \end{aligned}$$

$$
\begin{aligned}
N_{7,3}(q) = {} & q^{12} + q^{11} + 3 \cdot q^{10} + 5 \cdot q^9 + 9 \cdot q^8 + 13 \cdot q^7 + 22 \cdot q^6 + [-(q,2) + 34] \cdot q^5 \\
& + [-7 \cdot (q,2) + 65] \cdot q^4 + [-17 \cdot (q,2) + 1/2 \cdot (q,3) + 2 \cdot (q-1,3) + 215/2] \cdot q^3 \\
& + [-32 \cdot (q,2) - (q,3) + 5 \cdot (q-1,3) + 2 \cdot (q-1,4) + 166] \cdot q^2 \\
& + [-39 \cdot (q,2) - 1/2 \cdot (q,3) + 7 \cdot (q-1,3) + 4 \cdot (q-1,4) + 361/2] \cdot q \\
& + [-28 \cdot (q,2) - (q,3) + 2 \cdot (q-1,3) + 2 \cdot (q-1,4) + 121] \,.
\end{aligned}
$$

Theorem 1.7 *Let $q = p^s$ be an arbitrary prime power. Then*

$$
N_{5,4}(q) = q^2 + [-(q,2) + 7] \cdot q + [-6 \cdot (q,2) + 25] \,.
$$

$$
\begin{aligned}
N_{6,4}(q) = {} & q^{13} + q^{12} + 3 \cdot q^{11} + 4 \cdot q^{10} + 8 \cdot q^9 + 10 \cdot q^8 + [-(q,2) + 21] \cdot q^7 \\
& + [-3 \cdot (q,2) + 32] \cdot q^6 + [-8 \cdot (q,2) + 58] \cdot q^5 \\
& + [-16 \cdot (q,2) + 1/2 \cdot (q,3) + (q-1,3) + 181/2] \cdot q^4 \\
& + [-33 \cdot (q,2) + 3 \cdot (q-1,3) + 2 \cdot (q-1,4) + 156] \cdot q^3 \\
& + [-(q,2) \cdot (q-1,3) - 55 \cdot (q,2) + (q,3) + 9 \cdot (q-1,3) + 4 \cdot (q-1,4) + 229] \cdot q^2 \\
& + [-2 \cdot (q,2) \cdot (q-1,3) - 77 \cdot (q,2) \\
& \quad -1/2 \cdot (q,3) + 13 \cdot (q-1,3) + 6 \cdot (q-1,4) + 583/2] \cdot q \\
& + [-(q,2) \cdot (q-1,3) - 59 \cdot (q,2) - 1/2 \cdot (q,3) + 6 \cdot (q-1,3) + 4 \cdot (q-1,4) + 415/2] \,.
\end{aligned}
$$

Based on our computations, we propose the following conjecture on the degrees of the PORC functions $N_{d,r}(q)$.

Conjecture 1.8 *Let q be a prime power and let $r \in \mathbb{N}$.*

1. *$N_{r+1,r}(q) = N_{r+r^2-1,r}(q)$ is homogeneous of degree $\lfloor r/2 \rfloor$.*
2. *$N_{r+k,r}(q)$ is homogeneous of degree $1 + r^2k - (r^2 + k^2)$ for $k \in \{2, \ldots, r^2 - 2\}$.*

Chapter 2

PORC functions

2.1 Definition and historical background

In this section we investigate PORC functions, which were originally introduced by Higman [18]. Vaughan-Lee introduced many algorithmic tools for computing and determining PORC functions, see [34, 35, 36, 37]. We recall the definition of PORC functions from the introduction (Definition 1.2) and introduce the characteristic function of a residue class modulo m to shorten the notation of PORC functions.

Definition 2.1: ***Characteristic function of a residue class***
*Let $m \in \mathbb{N}$ and $l \in \{0, 1, \ldots, m-1\}$. We define $k = k_{l,m} : \mathbb{Z} \to \mathbb{Z}$ by $k_{l,m}(x) = 1$, if $x \equiv l \bmod m$, and $k_{l,m}(x) = 0$ otherwise. Thus, $k_{l,m}$ can be considered as the **characteristic function of the residue class of l modulo m**.*

It follows from Definition 1.2 that a PORC function f with modulus m can be written as

$$f(x) = \sum_{a=0}^{m-1} k_{a,m}(x) \cdot g_a(x), \tag{2.1}$$

where $k_{a,m}(x)$ are characteristic functions as in Definition 2.1. Vaughan-Lee formulates an algorithm for computing such characteristic functions, see [34]. We will consider this algorithm in its details in Section 2.2.

The idea of using PORC functions arises from group theory where the enumeration of groups of order p^n is still an open problem. Let $f_n(p)$ be the number of isomorphism types of p-groups of order p^n. Higman was one of the first people interested in finding the numbers $f_n(p)$. In his paper *Enumerating p-Groups I* [17], Higman gave bounds for the function $f_n(p)$. Then, he published *Enumerating p-Groups II* [18] and introduced the term *PORC function*. In this second paper, Higman also stated some algebraic structures whose numbers of isomorphism types are PORC. We refer to the paper of Higman [18] for the proofs.

Theorem 2.2 ***(Higman [18])***

1. *The number of isomorphism types of groups of order p^n whose Frattini subgroup is elementary abelian and central, considered as a function of the prime p for a fixed n, is PORC [18, Thm. 1.1.1].*

2. *The number of in-equivalent forms of degree r in m indeterminates over the field with q elements, considered as a function in q for fixed m and r, is PORC [18, Thm. 1.1.2].*

3. *The number of isomorphism types of arbitrary algebras of dimension n over the field with q elements, considered as a function in q for a fixed n, is PORC [18, Thm. 1.1.3].*

For instance, for dimensions $n \in \{1, 2, 3, 4\}$ the corresponding PORC functions in case of arbitrary algebras are computed by Vaughan-Lee [37]. These provide upper bounds for our results.

In case of the enumeration of p-groups PORC functions $f_n(p)$ are known for $n \leq 7$ up to now. However, for $n \geq 8$ it is still an open problem whether or not the functions $f_n(p)$ are PORC. Higman conjectured that the functions $f_n(p)$ are all PORC and this problem is well known as the PORC conjecture.

Conjecture 2.3 ***(Higman – PORC conjecture)***
The number $f_n(p)$ of isomorphism types of groups of order p^n, considered as a function in p, is PORC.

For $n \in \{1, 2, 3\}$ the PORC functions $f_n(p)$ are constant functions. In case of $n = 4$ one gets a PORC function with modulus 2: Depending on whether p is even or odd there are distinct constant polynomials. A complete list of isomorphism types of groups of order p^n for $1 \leq n \leq 4$ can be found in Burnside [5, Sections 112-118]. The corresponding enumerating functions $f_n(p)$ can be deduced.
For the case $n = 5$ Bagnera [1] published a list with isomorphism types of groups of order p^5, which was not free of errors, see [3, page 3]. By now, a complete list of isomorphism types of groups of order p^5 is known, see [2, 4, 20, 21, 32]. The corresponding functions $f_5(p)$ can be obtained using the cited authors' work.
The PORC functions $f_6(p)$ for the case $n = 6$ are determined by Newman, O'Brien and Vaughan-Lee [27]; for $n = 7$ the PORC functions considering the case of odd primes are computed by O'Brien and Vaughan-Lee [29]. For $p = 2$ and $n = 7$ the number $f_7(2)$ was determined by James, Newman, and O'Brien [22].
Du Sautoy and Vaughan-Lee [7] also worked on the case $n \geq 8$ and produced partial results. However, the general case still is an open problem.

2.2 The point-wise greatest common divisor

It is well known that $\mathbb{Q}[x]$ (the polynomial ring over $\mathbb{Q}$ with one indeterminate x) is a Euclidean ring. We are able to compute the greatest common divisor of two polynomials $f, g \in \mathbb{Q}[x]$ which is unique up to multiplication by units.
In this section we introduce an additional function defined on $\mathbb{Q}[x]$ called the ***point-wise greatest common divisor***.

Definition 2.4: ***Pointwise greatest common divisor***
Let $f, g \in \mathbb{Q}[x]$ be two polynomials with $f(\mathbb{Z}) \subseteq \mathbb{Z}$ and $g(\mathbb{Z}) \subseteq \mathbb{Z}$. We define a function

$$\mathrm{pgcd} : \mathbb{Z} \to \mathbb{Z}, \quad q \mapsto h(q) = \gcd(f(q), g(q)) \tag{2.2}$$

*and call this function the **point-wise greatest common divisor** of the polynomials f and g.*

Vaughan-Lee [34] showed that the pgcd is PORC and introduced an algorithm for its computation. The algorithm is essential for our work and, therefore, we include a description for completeness here.
The point-wise greatest common divisor shares many properties with the usual greatest common divisor. So, for three (or more) polynomials $f_1, f_2, f_3 \in \mathbb{Q}[x]$ we can compute their pgcd via

$$\mathrm{pgcd}(f_1, f_2, f_3) = \mathrm{pgcd}(f_1, \mathrm{pgcd}(f_2, f_3)). \tag{2.3}$$

It also holds that $\mathrm{pgcd}(f_1, f_2) = \mathrm{pgcd}(f_2, f_1)$.

Example 1. Let $f_1(q) = 2(q-1)$ and $f_2(q) = (q-1)(q+1)$. Then it follows that $f_1(\mathbb{Z}), f_2(\mathbb{Z}) \subseteq \mathbb{Z}$. Their greatest common divisor can be read off: $\gcd(f_1, f_2) = q - 1$. However, the pgcd is

$$\mathrm{pgcd}(f_1(q), f_2(q)) = \mathrm{pgcd}(2, q+1) \cdot (q-1) = \begin{cases} q-1, & q \text{ even}, \\ 2(q-1), & q \text{ odd}. \end{cases} \tag{2.4}$$

Thus, $\mathrm{pgcd}(f_1(q), f_2(q))$ is PORC. Using the characteristic functions (compare Definition 2.1) we obtain

$$\mathrm{pgcd}(f_1(q), f_2(q)) = 2 \cdot k_{1,2}(q) \cdot (q-1) + k_{0,2}(q) \cdot (q-1). \tag{2.5}$$

In Example 1 the pgcd of the given polynomials is PORC. Hence, the question arises, whether or not the pgcd can always be given as in Equation (2.4) respectively (2.5). We use results by Vaughan-Lee [34] (see Proposition 2.6 and Theorem 2.7) to proof that a pgcd of two polynomials is always PORC.

Proposition 2.5 *Given $l \in \mathbb{Z}$ and $m \in \mathbb{N}$. Then, the following function is PORC.*

$$h = h_{l,m} : \mathbb{Z} \to \mathbb{Z}, \quad q \mapsto \mathrm{pgcd}(q - l, m). \tag{2.6}$$

Proof: This observation directly follows from the fact that $\gcd(q + k \cdot m, m) = \gcd(q, m)$ for all $k \in \mathbb{Z}$. Therefore, $\gcd(q - l, m)$ is constant for all q in the same residue class modulo m. □

Proposition 2.6 ***(Vaughan-Lee [34])***
Let m, l, and $k_{l,m}(x)$ be as in Definition 2.1. Then, $k_{l,m}(x)$ can be expressed as a $\mathbb{Q}$-linear combination of terms of the form $\mathrm{pgcd}(x - l', m')$ for suitable l' and m'.

Proof: The proof is already given by Vaughan-Lee [34, Section 3] as part of the proof of [34, Theorem 2]. Since the proof is constructive it can be translated into an algorithm. For a better understanding of the algorithm we repeat Vaughan-Lee's proof here. At some points we add some calculations.
If $m = 1$ we only have one characteristic function $k_{0,1}(x) = 1$ for all x. Therefore, we assume $m > 1$. Let S be the set of prime factors of m. For every subset $T \subseteq S$ we define d_T to be the product of all elements in T. We then define a function

$$k : \mathbb{Z} \to \mathbb{Z}, \quad x \mapsto \sum_{T \subseteq S} (-1)^{|T|} \cdot \gcd\left(x, \frac{m}{d_T}\right). \tag{2.7}$$

The values of k only depend on the residue class of x modulo m: With Proposition 2.5 the value of every summand in Equation (2.7) only depends on the residue class of x modulo m/d_T. Because of $\gcd(x, m/d_T) = \gcd(x \cdot d_T, m)/d_T$ the value of every summand in Equation (2.7) is uniquely determined by the residue class of x modulo m.
The following calculation shows $k(m) \neq 0$. Assume $S = \{p_1, \ldots, p_s\}$. Thus,

$$\begin{aligned} k(m) &= \sum_{T\subseteq S} (-1)^{|T|} \cdot \gcd\left(m, \tfrac{m}{d_T}\right) = \sum_{T\subseteq S} (-1)^{|T|} \cdot \tfrac{m}{d_T} \\ &= m \sum_{T\subseteq S} (-1)^{|T|} \cdot \prod_{p\in T} p^{-1} = m \prod_{p\in S} p^{-1} \sum_{T\subseteq S} (-1)^{|T|} \prod_{p\in S\setminus T} p \\ &= m \prod_{p\in S} p^{-1} \prod_{p\in S} (p-1) = m \prod_{p\in S} \tfrac{p-1}{p} \neq 0. \end{aligned} \tag{2.8}$$

Now, we consider x with $1 \le x < m$ and we choose $p \in S$ such that the power of p dividing x is strictly less than the power of p dividing m. We define $U = S \setminus \{p\}$ and obtain

$$k(x) = \sum_{T\subseteq U} (-1)^{|T|} \left(\gcd\left(x, \tfrac{m}{d_T}\right) - \gcd\left(x, \tfrac{m}{pd_T}\right)\right) = 0. \tag{2.9}$$

At this step we use that for the chosen p we have $\gcd(x, \frac{m}{d_T}) = \gcd(x, \frac{m}{pd_T})$ for all $T \subseteq U$. The equality amongst the $\gcd$-terms can be seen as follows: We write $m = p^e \tilde{m}$ where e is chosen maximal such that $p^e \mid m$. Analogously we choose f maximal such that $p^f \mid x$. We write $x = p^f \tilde{x}$ and have $p \nmid \tilde{m}, \tilde{x}, d_T$. By construction it is $f \lneq e$. This leads to

$$\gcd\left(x, \frac{m}{d_T}\right) = \gcd\left(p^f \tilde{x}, \frac{p^e \tilde{m}}{d_T}\right) \overset{f\lneq e}{=} \gcd\left(p^f \tilde{x}, \frac{p^{e-1}\tilde{m}}{d_T}\right) = \gcd\left(p^f \tilde{x}, \frac{p^e \tilde{m}}{pd_T}\right) = \gcd\left(x, \frac{m}{pd_T}\right). \tag{2.10}$$

We have constructed a function that is only non-zero when evaluated at $x \in m\mathbb{Z}$. Defining $c = k(m)$ we find

$$\frac{1}{c} \cdot k(x-l) = \begin{cases} 1, & x \equiv l \bmod m, \\ 0, & \text{otherwise.} \end{cases} \tag{2.11}$$

So, $\frac{1}{c} \cdot k(x-l)$ is the characteristic function $k_{l,m}$ for the residue class of l modulo m. □

Example 2. Let us compute the characteristic functions $k_{1,2}(q)$ and $k_{0,2}(q)$. In both cases we have $m = 2$ and, hence, it is $S = \{2\}$. We define the function k via

$$k : \mathbb{Z} \to \mathbb{Z},\ q \mapsto \sum_{T\subseteq\{2\}} (-1)^{|T|} \cdot \gcd\left(q, \tfrac{2}{d_T}\right) = \gcd(q,2) - \gcd(q,1) \tag{2.12}$$

and get $c = k(2) = 1$. Then, we compute the characteristic functions as follows. Note that $\gcd(q,1) = 1$ for all $q \in \mathbb{Z}$. Thus,

$$k_{1,2}(q) = k(q-1) = \gcd(q-1,2) - \gcd(q-1,1) = \gcd(q-1,2) - 1, \tag{2.13}$$
$$k_{0,2}(q) = k(q-2) = \gcd(q-2,2) - \gcd(q-2,1) = \gcd(q,2) - 1. \tag{2.14}$$

Theorem 2.7 ***(Vaughan-Lee [34, Theorem 2])***
The point-wise greatest common divisor of a set of integer polynomials $\{f_1, \ldots, f_s\}$ can be expressed in the form $d \cdot f$ where f is an integer polynomial and where

$$d = \alpha + \sum_{i=1}^{r} \alpha_i \gcd(q - n_i, m_i) \tag{2.15}$$

for some $\alpha, \alpha_1, \ldots, \alpha_r \in \mathbb{Q}$, some $m_1, \ldots, m_r \in \mathbb{N}$, and some $n_1, \ldots, n_r \in \mathbb{Z}$ with $0 \le n_i < m_i$ for all i.

Proof: Again, we follow Vaughan-Lee [34, Chapter 3] and repeat his constructive proof since parts of our algorithm are based on this theorem. Let $f_1, f_2, \ldots, f_s \in \mathbb{Z}[q]$ and let $h = \mathrm{pgcd}(f_1(q), f_2(q), \ldots, f_s(q))$. We give an algorithm for the computation of h.
We start by computing the greatest common divisor f of $f_1, \ldots, f_s$ over the Euclidean ring $\mathbb{Q}[x]$ and choose it to be primitive over $\mathbb{Z}$. Thus all coefficients of f are integers and the greatest common divisor over all coefficients is 1. The Euclidean algorithm further gives polynomials $g_1, \ldots, g_s \in \mathbb{Q}[x]$ such that

$$f_1 g_1 + f_2 g_2 + \ldots + f_s g_s = f. \tag{2.16}$$

We assume that all coefficients of the g_i are of the form $\frac{a}{b}$ with $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $\gcd(a, b) = 1$. Let m be the least common multiple of all denominators of the coefficients of the polynomials g_i. It follows that $h(q) = d(q) \cdot f(q)$ for all $q \in \mathbb{Z}$ and some function d such that $d(q)$ divides m depending on the residue class of q modulo m. Thus, h must be a PORC function with modulus m.
So, it remains to compute the value of d for every residue class modulo m. We do this by choosing for every residue class a representative l' with $l' \ge 0$ and l' minimal such that $f_j(l') \ne 0$ for $1 \le j \le s$. We compute the value $d = d(l')$ as follows:

$$d(l') = \frac{\gcd(f_1(l'), \ldots, f_s(l'))}{f(l')}. \tag{2.17}$$

Now, we find the function $d(x)$ to be

$$d(x) = \sum_{l=0}^{m-1} d(l') \cdot k_{l,m}(x) \qquad \text{with} \qquad l' \equiv l \bmod m \tag{2.18}$$

and with $k_{l,m}(x)$ as in Proposition 2.6. Rearranging the terms gives the form of d as in the theorem. □

Example 3. Again, we consider the rational polynomials $f_1(q) = 2(q-1)$ and $f_2(q) = (q-1)(q+1)$ from Example 1. This time we compute $\mathrm{pgcd}(f_1, f_2)$ following the algorithm given in the proof of Theorem 2.7. Over the Euclidean domain $\mathbb{Q}[q]$ we get

$$f(q) := \gcd(f_1, f_2) = q - 1 = \tfrac{1}{2} \cdot f_1 + 0 \cdot f_2. \tag{2.19}$$

Thus, we have $\mathrm{pgcd}(f_1, f_2) = d(q) \cdot (q-1)$ and $m = 2$. For the residue class $2\mathbb{Z}$ we choose the representative 2, for the class $1 + 2\mathbb{Z}$ we choose the representative 3. We get

$$d(3) = \frac{\gcd(f_1(3), f_2(3))}{f(3)} = \frac{\gcd(4, 8)}{2} = 2, \tag{2.20}$$

$$d(2) = \frac{\gcd(f_1(2), f_2(2))}{f(2)} = \frac{\gcd(2, 3)}{1} = 1. \tag{2.21}$$

We obtain the function d as

$$d(q) = 2 \cdot k_{1,2}(q) + 1 \cdot k_{0,2}(q). \tag{2.22}$$

Hence, the pgcd agrees with our computation above, compare Equation (2.5):

$$\mathrm{pgcd}(f_1, f_2) = d(q) \cdot f(q) = 2 \cdot k_{1,2}(q) \cdot (q-1) + k_{0,2}(q) \cdot (q-1). \tag{2.23}$$

2.3 Algorithmic computation of a pgcd

In the previous section we have proved that a point-wise greatest common divisor of polynomials is a PORC function. The constructive proofs of Proposition 2.6 and Theorem 2.7 can be used to develop an algorithm. A first function (compare Algorithm 2 in Appendix B) is based on Proposition 2.6 and computes all characteristic functions $k_{l,m}$ for a given m. A second function (see Algorithm 3 in Appendix B) is based on Theorem 2.7 and computes the point-wise greatest common divisor of a given set of polynomials.
We encounter a problem when computing PORC functions since there is no uniqueness in writing them. The following Example 4 illustrates this problem.

Example 4. We consider the two PORC functions f and g given as follows. Recall that round brackets $(,)$ denote the greatest common divisor, square brackets are used for grouping terms together.

$$\begin{aligned} f(q) = {} & [(q-1,2)-1] \cdot q^2 + [(q,3)+(q-1,3)+(q-1,2)+1] \cdot q \\ & + [(q,3)+(q-1,3)+2], \end{aligned} \tag{2.24}$$

$$\begin{aligned} g(q) = {} & [2-(q,2)] \cdot q^2 + [3+(q,2)+2 \cdot (q-1,2)-(q-2,3)] \cdot q \\ & + [1+2 \cdot (q,2)+2 \cdot (q-1,2)-(q-2,3)]. \end{aligned} \tag{2.25}$$

Even though f and g seem to be different as all terms involving greatest common divisors are different, as functions we have $f = g$. How is it possible to recognise such incidents?

Lemma 2.8 *Let $n \in \mathbb{N}$ with $n = \prod_{i=1}^{k} p_i^{e_i}$ with distinct primes p_i. Then, for every $a \in \mathbb{Z}$ it is*

$$\gcd(a, n) = \prod_{i=1}^{k} \gcd(a, p_i^{e_i}). \tag{2.26}$$

Proof: This follows immediately from the fact that $\gcd(a, n \cdot m) = \gcd(a, n) \cdot \gcd(a, m)$ for numbers $n, m \in \mathbb{Z}$ with $\gcd(n, m) = 1$. □

Lemma 2.9 *Given $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ then for all $k \in \mathbb{Z}$ there exists a constant $c \in \mathbb{N}$ such that*

$$\sum_{i=k}^{k+n-1} \gcd(a - i, n) = c. \tag{2.27}$$

Proof: Without loss of generality we can choose $k = 0$ in Equation (2.27) since the expression $\gcd(a - i, n)$ is n-periodic as a function in i. Furthermore, the value of $\gcd(a - i, n)$ only depends on the residue class of $a - i$ modulo n. Hence, we obtain the value c by summing up the numbers $\gcd(a - i, n)$ for the given a and n. □

We are now able to formulate rules on how we use gcd's inside PORC functions.

- The modulus of the gcd's shall be a prime power: Lemma 2.8 states that we can decompose $\gcd(q - i, n)$ into a product $\gcd(q - i, p_1^{e_1}) \cdots \gcd(q - i, p_r^{e_r})$ where $p_1, \ldots, p_r$ are the distinct prime factors of n with their powers $e_1, \ldots, e_r$.
- We only use gcd's of the form $\gcd(q - i, n)$ for $0 \leq i \leq n - 2$. This follows from Lemma 2.9.

Example 4 (continued). Recall the PORC functions defined in Equations (2.24) and (2.25). First, we recognise that only gcd's with primes 2 and 3 occur. So, there is no need to apply the idea of Lemma 2.8 here. For the two occurring primes we find

$$\sum_{i=0}^{1} \gcd(q - i, 2) = 3 \quad \text{and} \quad \sum_{i=0}^{2} \gcd(q - i, 3) = 5. \tag{2.28}$$

This allows us to replace $\gcd(q - 1, 2)$ and $\gcd(q - 2, 3)$ and we get

$$\begin{aligned} f(q) = g(q) = [2 - (q, 2)] \cdot q^2 + [(q, 3) + (q - 1, 3) - (q, 2) + 4] \cdot q \\ + [(q, 3) + (q - 1, 3) + 2] . \end{aligned} \tag{2.29}$$

Note that there is no unique presentation guaranteed for PORC functions in general even after applying our rules. But these rules help to manage the huge amount of gcd's inside the functions.
Whenever we need to test whether or not two PORC functions, say $f(q)$ and $g(q)$, are identical (as functions) we first apply the rules on the gcd's. Then, we consider the difference $d(q) = f(q) - g(q)$. Let m be the modulus of the PORC function d. We consider $d(q) = a_n q^n + \ldots + a_1 q + a_0$ as a polynomial in q whose coefficients a_i are $\mathbb{Q}$-linear combinations of gcd's. By construction the coefficients are periodic functions in q with period m. We evaluate every coefficient a_i at x for every $0 \leq q < m$. If $a_i(q) = 0$ for all q we can consider a_i to be constant zero. If all coefficients are zero functions the difference d must be zero for every q and we can consider $f(q)$ and $g(q)$ to be equal. In our algorithm this procedure is called ***simplification***.

Example 5. Now, let us consider the PORC functions

$$f(q) = [(q-1,4) + (q-2,4) + 1] \cdot q + [4 \cdot (q,2) - (q,4) + (q-1,4) - 1]\,, \tag{2.30}$$
$$g(q) = [9 - (q,4) - (q-3,4)] \cdot q + [(q-1,4) + (q-2,4) + 1]\,. \tag{2.31}$$

Applying our rules to the given functions and computing the difference $f(q) - g(q)$ yields:

$$f(q) - g(q) = 4 \cdot (q,2) - (q,4) - (q-2,4) - 2. \tag{2.32}$$

We consider the difference as a constant polynomial in q and the coefficient of the constant term is $a_0 = 4 \cdot (q,2) - (q,4) - (q-2,4) - 2$.
There are only $\gcd$'s with 2 and 4 and their total modulus is $l = \operatorname{lcm}(2,4) = 4$. We check $a_0(0) = 0$, $a_0(1) = 0$, $a_0(2) = 0$, and $a_0(3) = 0$ which gives $a_0(q) = 0$ for all q. Hence, it holds $f(q) - g(q) = 0$.

Chapter 3

Types of matrices

3.1 Partitions

The theory of partitions serves as a tool to describe the so-called types of matrices (see Section 3.2). In this section we will follow the introduction to partitions in [24, Chapter 1].

Definition 3.1: ***Partition***
*A **partition** is a finite sequence*

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \tag{3.1}$$

of non-negative integers in non-increasing order (hence, we have $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$).

We do not distinguish between partitions that only differ by zeros at the end, so for example we consider $(2,1,0,0)$, $(2,1,0)$, or $(2,1)$ as equal. We call the non-zero elements λ_i in Equation (3.1) the ***parts*** of λ. The number of non-zero parts is the ***length*** of λ. Further, we define the ***weight*** of λ to be

$$|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_r. \tag{3.2}$$

Let $n \in \mathbb{N}$ be a number with $|\lambda| = n$, then we say that λ is a partition of n.
The ***multiplicity*** m_i of a part i in λ is defined as

$$m_i = m_i(\lambda) = |\{\, \lambda_j \mid \lambda_j = i \,\}|. \tag{3.3}$$

Sometimes it is helpful to arrange a partition in a ***diagram***. Given a partition λ we draw the diagram as follows: Each part λ_i of λ is represented by a row consisting of λ_i dots. The row belonging to λ_i is the i'th row in the diagram. All those dots are arranged in a square grid aligned to the left. Those diagrams are also called Young-diagrams or Ferrers-diagrams.
A diagram can be used to define the ***conjugate partition*** λ' of λ. The conjugate partition λ' is the partition we obtain as the partition belonging to the transpose of the diagram of λ. By transposing we mean reflecting the diagram along the main diagonal. The parts of λ' are denoted by λ'_i.

Using the conjugate partition we are able to give a formula for the multiplicities $m_i(\lambda)$. Since it can be read off from a partition's diagram (compare Figure 3.1 (a)) we note:

$$\lambda'_i = |\{\, j \mid \lambda_j \geq i \,\}|. \tag{3.4}$$

This yields

$$m_i(\lambda) = \lambda'_i - \lambda'_{i+1}. \tag{3.5}$$

Let us introduce the function

$$n(\lambda) = \sum_{i\geq 1} (i-1)\lambda_i. \tag{3.6}$$

Lemma 3.2 *Let λ be a partition with conjugate λ'. Then,*

$$n(\lambda) = \sum_{i\geq 1} \binom{\lambda'_i}{2}. \tag{3.7}$$

Proof: Let us start with $n(\lambda)$ by its definition. If we replace the dots in the i'th row of the diagram of λ by the number $i-1$ and if we then sum up all those numbers we will get the value $n(\lambda)$. Let us now read these numbers along the columns: In the first column we have the numbers 0 to $\lambda'_1 - 1$. Summing them up gives $\frac{1}{2} \cdot \lambda'_1(\lambda'_1 - 1)$. Analogously, we obtain that the numbers in column number i add up to $\frac{1}{2} \cdot \lambda'_i(\lambda'_i - 1)$. Rearranging the terms, we find

$$\frac{\lambda'_i(\lambda'_i - 1)}{2} = \frac{\lambda'_i!}{(\lambda'_i - 2)! \cdot 2!} = \binom{\lambda'_i}{2}. \tag{3.8}$$

We are now left to add up all those sums and gain the wanted result. □

We will close this section with a technical lemma which will be used as an estimate for the degrees of polynomials (compare Lemma 3.10).

Lemma 3.3 *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \ldots \geq \lambda_k > 0$ be a partition. Then*

$$\sum_{i\geq 1} (2i-1)\lambda_i \geq \sum_{i\geq 1} \frac{m_i(\lambda) \cdot (m_i(\lambda)+1)}{2}. \tag{3.9}$$

Proof: First, we rewrite the left-hand-side using the function $n(\lambda)$.

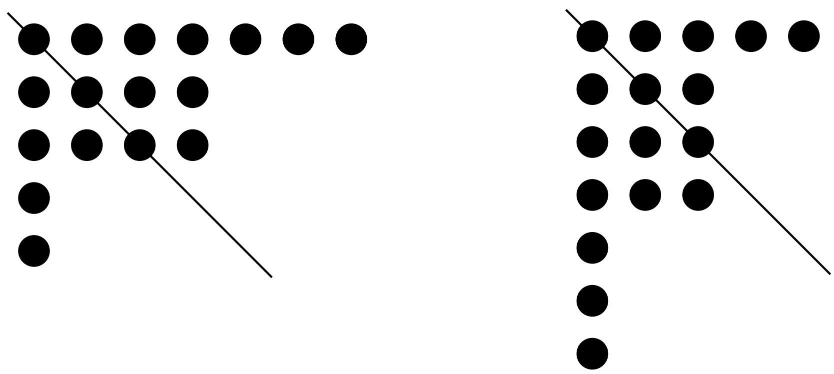
$$\begin{aligned} \sum_{i\geq 1} (2i-1)\lambda_i &= 2\sum_{i\geq 1}(i-1)\lambda_i + \sum_{i\geq 1}\lambda_i = 2n(\lambda) + |\lambda| \\ &= 2\sum_{i\geq 1}\binom{\lambda'_i}{2} + \sum_{i\geq 1}\lambda'_i = 2\sum_{i\geq 1}\frac{\lambda'_i(\lambda'_i-1)}{2} + \sum_{i\geq 1}\lambda'_i \\ &= \sum_{i\geq 1}\lambda'^2_i - \lambda'_i + \sum_{i\geq 1}\lambda'_i = \sum_{i\geq 1}\lambda'^2_i. \end{aligned} \tag{3.10}$$

Now, we will go on with the right-hand-side and remember that $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$.

$$\begin{aligned}
\sum_{i\geq 1} \frac{(m_i(\lambda)+1)\cdot m_i(\lambda)}{2} &= \sum_{i\geq 1} \frac{(\lambda'_i - \lambda'_{i+1}+1)(\lambda'_i - \lambda'_{i+1})}{2} \\
&= \sum_{i\geq 1} \lambda'^2_i + \frac{1}{2}\sum_{i\geq 1}\left(\lambda'^2_{i+1} - \lambda'^2_i + \lambda'_i - \lambda'_{i+1} - 2\lambda'_i\lambda'_{i+1}\right) \\
&= \sum_{i\geq 1} \lambda'^2_i + \frac{1}{2}\sum_{i\geq 1}\left(\lambda'^2_{i+1} - \lambda'^2_i - (\lambda'_{i+1} - \lambda'_i)\right) - \underbrace{\sum_{i\geq 1}\lambda'_i\lambda'_{i+1}}_{\geq 0} \qquad (3.11) \\
&\leq \sum_{i\geq 1} \lambda'^2_i + \frac{1}{2}\sum_{i\geq 1}\left((\lambda'_{i+1} - \lambda'_i)(\lambda'_{i+1} + \lambda'_i - 1)\right) \\
&\leq \sum_{i\geq 1} \lambda'^2_i.
\end{aligned}$$

The last inequality holds because of $\lambda'_{i+1} - \lambda'_i \leq 0$ and $\lambda'_{i+1} + \lambda'_i \geq 1$. Then, the lemma holds. □

Example 6. We consider $\lambda = (7,4,4,1,1)$. The length of λ is 5 and its weight is $|\lambda| = 17$. We further have $m_1(\lambda) = 2$, $m_4(\lambda) = 2$ and $m_7(\lambda) = 1$. It is $m_i(\lambda) = 0$ for all $i \notin \{1,4,7\}$. Its diagram is given in Figure 3.1 (a). Reflecting the diagram of λ yields Figure 3.1 (b) from which we can read off $\lambda' = (5,3,3,3,1,1,1)$. We have $n(\lambda) = 19$.



(a) The diagram of the partition $\lambda = (7,4,4,1,1)$.

(b) The diagram of λ' is obtained by reflecting Figure (a).

Figure 3.1: The diagram of $\lambda = (7,4,4,1,1)$ and its conjugate partition $\lambda' = (5,3,3,3,1,1,1)$.

3.2 The type of a matrix

This section is dedicated to summarise information on the ***rational canonical form*** of a matrix and to introduce the ***type*** of a matrix. A formal introduction into the topic of rational canonical forms for modules in general can be found in [8, Chapter 12]. Since we will only consider matrices defined over fields we restrict our definition to this case.
Recall that two matrices $A, B \in \mathbb{F}^{n \times n}$ are called similar, denoted by $A \sim B$, if we can find an invertible matrix $C \in \mathbb{F}^{n \times n}$ such that $A = CBC^{-1}$.

Definition 3.4: ***Companion matrix***
Let $\mathbb{F}$ be a field and $a(x) = x^k + b_{k-1}x^{k-1} + \ldots + b_1 x + b_0 \in \mathbb{F}[x]$ be a monic polynomial. The companion matrix of $a(x)$, denoted by $\mathcal{C}_{a(x)}$, is the $k \times k$-matrix

$$\mathcal{C}_{a(x)} = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & -b_0 \\ 1 & 0 & \dots & \dots & 0 & -b_1 \\ 0 & 1 & \dots & \dots & 0 & -b_2 \\ 0 & 0 & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \dots & 1 & -b_{k-1} \end{pmatrix}. \tag{3.12}$$

Let $\mathbb{F}$ be a field and $A \in \mathbb{F}^{n \times n}$ be a matrix. Then, there exist monic polynomials $a_1, \ldots, a_m \in \mathbb{F}[x]$ called the ***invariant factors*** of A such that the following holds (compare [8, Chapter 12, Theorem 4]):

- There is a similarity

$$A \sim \mathcal{C}_{a_1(x)} \oplus \mathcal{C}_{a_2(x)} \oplus \ldots \oplus \mathcal{C}_{a_m(x)}. \tag{3.13}$$

- The polynomials $a_1(x), \ldots, a_m(x) \in \mathbb{F}[x]$ satisfy the divisibility chain

$$a_1(x) \mid a_2(x) \mid \ldots \mid a_m(x). \tag{3.14}$$

- The polynomial $a_m(x)$ is the minimal polynomial of A.

Definition 3.5: ***Rational canonical form of a matrix***
A matrix A is said to be in rational canonical form if it is the direct sum of companion matrices for non-constant monic polynomials $a_1(x), \ldots, a_m(x)$ with $a_1(x) \mid \ldots \mid a_m(x)$ such that $a_m(x)$ is the minimal polynomial of A.

Theorem 3.6 ***(compare [8, Chap. 12, Theorem 17])***
Let A and B be two $n \times n$-matrices over the field $\mathbb{F}$. Then A and B are similar if and only if A and B have the same rational canonical form.

The invariant factors are not necessarily irreducible. Hence, we can decompose every $a_i(x)$ into the product of powers of monic irreducible polynomials:

$$a_i(x) = p_1(x)^{\beta_{i,1}} \cdot p_2(x)^{\beta_{i,2}} \cdot \ldots \cdot p_s(x)^{\beta_{i,s}}. \tag{3.15}$$

Some of the exponents $\beta_{i,j}$ in Equation (3.15) might be 0.

Definition 3.7: ***Elementary divisors***
Let $A \in \mathbb{F}^{n \times n}$ be a matrix with invariant factors $a_1(x), \ldots, a_m(x)$. Let every $a_i(x)$ be decomposed into a product of monic irreducible polynomials as in Equation (3.15). The polynomials $p_j^{\beta_{i,j}}$ (including their multiplicities) such that $\beta_{i,j} \neq 0$ are the elementary divisors of A.

For every polynomial p_j in Equation (3.15) we define the non-increasing finite sequence

$$s_j := (\beta_{m,j}, \beta_{m-1,j}, \ldots) \tag{3.16}$$

consisting of the non-zero terms $\beta_{i,j}$. For the considered matrix A it follows

$$A \sim \bigoplus_{j=1}^{s} \bigoplus_{\beta \in s_j} \mathcal{C}_{p_j^\beta(x)}. \tag{3.17}$$

Let $d_j = \deg(p_j)$ for every polynomial p_j from Equation (3.15). We define for the matrix A the following lexicographically sorted list:

$$t := t_A := ((d_1, s_1), (d_2, s_2), \ldots, (d_m, s_m)). \tag{3.18}$$

Definition 3.8: ***Type of a matrix***
*Given an $n \times n$-matrix over a field $\mathbb{F}$. Using the above notation the lexicographically ordered list $t = t_A$ as in Equation (3.18) is called the **type** of A.*
*A pair (d_i, s_i) is called a **type-parameter** of the type t.*

We close this section by determining the number $\tau(n)$ of types of matrices of a given matrix group $\mathrm{GL}(n, q)$. A generating function for $\tau(n)$ can be found in [16]. This function is given via

$$\sum_{n=0}^{\infty} \tau(n) x^n = \prod_{i=1}^{\infty} \mathfrak{p}(x^i)^{\mathfrak{p}_i}. \tag{3.19}$$

In the above formula $\mathfrak{p}(x) = \prod_{i=1}^{\infty}(1 - x^i)^{-1} = \sum_{n=0}^{\infty} \mathfrak{p}_n x^n$ is the partition function. The coefficients $\mathfrak{p}_n$ are the coefficients of its series expansion and they give the number of partitions of the integer n. The first few values for $\tau(n)$ are given in Table 3.1 (compare [33]).

n	1	2	3	4	5	6	7	8	9	10
$\tau(n)$	1	4	8	22	42	103	199	441	859	1784

Table 3.1: The first few values $\tau(n)$ of types of matrices in $\mathrm{GL}(n, q)$.

3.3 Properties of types

So far, we introduced the type of any square matrix. We will now specialise to matrices in $\mathrm{GL}(n, \mathbb{F})$. Furthermore, we are interested in finite fields and, hence, we have $\mathbb{F} = \mathbb{F}_q$ with any prime-power q. We then omit the symbol "$\mathbb{F}$" for the field when denoting the general linear group. We simply use $\mathrm{GL}(n, q)$. This is a finite group and its order is

$$|\mathrm{GL}(n,q)| = \prod_{i=1}^{n}(q^n - q^{n-i}) = (q^n - 1)(q^n - q) \cdot \ldots \cdot (q^n - q^{n-1}). \tag{3.20}$$

For any set S of square matrices over a field, let $\mathcal{T}(S)$ be the set of all types of matrices in S. We will only work with $S = \mathrm{GL}(n, q)$. Thus, we describe some properties of matrices of $\mathrm{GL}(n, q)$ having the same type. Let us introduce some notation.
Let t be a type of matrices in $\mathrm{GL}(n, q)$ and let $\mathcal{E}(t)$ be the set of all elements in $\mathrm{GL}(n, q)$ of type t. By construction, $\mathcal{E}(t)$ is a union of conjugacy classes of $\mathrm{GL}(n, q)$.

Lemma 3.9 *Let G be a finite group and let $a \in G$. Let $cc(a) = \{sas^{-1} \mid s \in G\}$ be the conjugacy class of a and let $C_G(a) := \{g \in G \mid gag^{-1} = a\}$ be the centraliser of a in G. Then, there exists a bijection between $cc(a)$ and the set of cosets $G/C_G(a)$.*

Proof: Use [23, Proposition 5.1, p. 28] in the case that G acts on itself by conjugation. □

In our case Lemma 3.9 yields: Let $g \in \mathrm{GL}(n, q)$ and let $cc(g)$ be its conjugacy class in $\mathrm{GL}(n, q)$. Then, Lagrange's Theorem (see for example [14, Satz 1.10.3.]) yields

$$|cc(g)| = \left|\mathrm{GL}(n,q)/C_{\mathrm{GL}(n,q)}(g)\right| = \frac{|\mathrm{GL}(n,q)|}{|C_{\mathrm{GL}(n,q)}(g)|}. \tag{3.21}$$

We would like to obtain a formulation of Equation (3.21) independent of the chosen matrix g. Let c_t be the order of the centraliser $C_{\mathrm{GL}(n,q)}(g)$ of an element g of type t in $\mathrm{GL}(n, q)$. Further, let k_t be the number of distinct conjugacy classes of matrices in $\mathcal{E}(t)$.

Lemma 3.10 *Let t be a type of a matrix in $\mathrm{GL}(n, q)$. Then:*

1. *The order c_t of the centraliser of an element of type t in $\mathrm{GL}(n, q)$ is invariant of the type.*
2. *For a fixed type t the order $c_t = c_t(q)$, considered as a function in q, is an integer polynomial.*
3. *The set $\mathcal{E}(t)$ consists of k_t distinct conjugacy classes and for a fixed type t the number $k_t = k_t(q)$, considered as a function in q, is an integer polynomial.*
4. *The number of elements of the set $\mathcal{E}(t)$, considered as function in q, is a rational polynomial and it is $|\mathcal{E}(t)| = k_t \cdot |\mathrm{GL}(n, q)|/c_t$.*

Proof:

1. The first part follows from [16, Lemma 2.1] or [24, p. 181]. The authors identify an element $g \in \mathrm{GL}(n,q)$ with its action on the $\mathbb{F}_q[x]$-module $\mathbb{F}_q^n$. It is shown that for all matrices of the same type the module $\mathbb{F}_q^n$ splits into similar submodules. The order c_t of the centraliser just depends on this decomposition and hence does not depend on the chosen matrix g of type t.

2. We obtain the order c_t of the centraliser of an element of type $t = ((n_1, s_1), \ldots, (n_m, s_m))$ again using [16, Lemma 2.1] or [24, p. 181]. The authors give the order of the centraliser c_j for every block matrix belonging to a type-parameter (n_j, s_j). Then, one obtains c_t as the product over all c_j's. In detail:

$$c_t = \prod_{j=1}^{m} c_j \qquad \text{with} \qquad c_j = c_j(q) = q^{f(s_j)} \cdot \prod_{i \geq 1} \varphi_{m_i(s_j)}(q^{-1}). \tag{3.22}$$

   In the above formula it is $\varphi_k(x) = (1-x) \cdot (1-x^2) \cdot \ldots \cdot (1-x^k)$. We recall that $m_i(s_j)$ states how many times the number i appears within the partition s_j (compare Section 3.1). The function f is a partition-valued function mapping into the integers via $f(s_j) = \sum_{i \geq 1} (2i-1) s_{i,j}$ with $s_j = (s_{1,j}, s_{2,j}, \ldots, s_{m_j,j})$. We refer to [16, Lemma 2.1] or [24, p. 181] for the proof that Equation (3.22) gives the order of the centraliser.

   It remains to be shown that c_j is an integer polynomial. First we consider the right product in Equation (3.22). Replacing q^{-1} by x allows us to consider $\varphi_{m_i(s_j)}$ as a polynomial in x. As the degree of $\varphi_k(x)$ in x is given by the sum of all numbers from one to k, we find

$$\deg\left(\prod_{i \geq 1} \varphi_{m_i(s_j)}(x)\right) = \sum_{i \geq 1} \sum_{k=1}^{m_i(s_j)} k = \sum_{i \geq 1} \frac{m_i(s_j)(m_i(s_j)+1)}{2} =: D. \tag{3.23}$$

   Replacing x by q^{-1} again and expanding the right product in Equation (3.22), the term with the smallest exponent is q^{-D}. Now, we apply Lemma 3.3 which states that $f(s_j) \geq D$. So, all negative exponents of the expanded form will be cancelled to some positive ones when multiplying the product by $q^{f(s_j)}$. Hence, every c_j can be considered as an integer polynomial in q. As a product over several $c_j(q)$, the centraliser order c_t can be considered as an integer polynomial as well.

3. We obtained the type of a matrix as a generalisation of the rational canonical form. Hence, $\mathcal{E}(t)$ is the disjoint union of several conjugacy classes. We need to determine the number of conjugacy classes of elements of type t.

   As usual, let $t = ((n_1, s_1), \ldots, (n_m, s_m))$. So, each type-parameter (n_j, s_j) can be represented by a generalised companion matrix of an irreducible polynomial of degree n_j. The term "generalised" means the following: Let p be an irreducible polynomial of degree n_i. For $s_j = s_{1,j}, s_{2,j}, \ldots, s_{m_j,j}$ we define the companion matrices $\mathcal{C}_{p^{s_{1,j}}(x)}, \ldots, \mathcal{C}_{p^{s_{m_j,j}}(x)}$. Then, the generalised companion matrix is defined to be the block diagonal matrix whose blocks are the matrices $\mathcal{C}_{p^{s_{1,j}}(x)}, \ldots, \mathcal{C}_{p^{s_{m_j,j}}(x)}$.

   We need to find the number of irreducible polynomials of a given degree. Let μ denote the Möbius function, then the number $\nu_d(q)$ of irreducible polynomials of degree d over the field with q elements is given by (compare [19, 4.13, Corollary 2])

$$\nu_d(q) = \frac{1}{d} \sum_{f \mid d} \mu\left(\tfrac{d}{f}\right) q^f. \tag{3.24}$$

If we further define $y_i = |\{l \in \{1, \ldots, i-1\} \mid n_l = n_i\}|$ the number k_t is given by

$$k_t = \frac{1}{m_1! \cdot \ldots \cdot m_w!} \cdot \prod_{i=1}^{m} (\nu_{n_i}(q) - y_i) \,. \tag{3.25}$$

The numbers m_k in formula (3.25) give the multiplicities of the type-parameters (n_j, s_j) within t. So, we have that $\frac{1}{m_1! \cdot \ldots \cdot m_w!}$ is the number of permutations of the type-parameters of t which do not change the type. We will abbreviate this number by $\pi(t)$.

The function $q \mapsto k_t(q)$ can be considered as a polynomial in $\mathbb{Q}[q]$. We need to show that all coefficients of this function are integers.

We agreed to sort the type-parameters (n_j, s_j) of t lexicographically. Whenever two pairs (n_j, s_j) are equal, they occur next to each other in t. Without loss of generality, let $(n_1, s_1) = \ldots = (n_k, s_k)$. Hence, we have $m_1 = k$. Let $\nu = \nu_{n_1}(q)$. The first k factors within the product sign of formula (3.25) together with $m_1 = k$ yield

$$\frac{1}{k!} \cdot ((\nu - y_1) \cdot \ldots \cdot (\nu - y_1 - (k-1))) = \frac{1}{k!} \cdot \frac{(\nu - y_1)!}{(\nu - y_1 - k)!} = \binom{\nu - y_1}{k} \in \mathbb{N}. \tag{3.26}$$

If we do the same with all following products we are able to cancel out all the terms $\frac{1}{m_j!}$ and we are left with only integer coefficients.

4. We consider k_t, G and c_t as polynomials in q. With c_t being a divisor of G for every q we find that $k_t(q) \cdot G/c_t(q)$ is a polynomial, too. Lemma 3.9 states that G/c_t denotes the size of the conjugacy class of an element of type t. With k_t being the number of distinct conjugacy classes of type t we obtain the last part of the lemma.

□

The following lemma yields a further invariant for matrices of the same type.

Lemma 3.11 ***(see Eick and O'Brien [10, Section 4.2])***
Let t be a type of matrices of $\mathrm{GL}(n,q)$ and let $k \in \mathbb{N}$. Then there exists a polynomial $f_{t,k}(q) \in \mathbb{Z}[q]$ such that every element of type t in $\mathrm{GL}(n,q)$ has $f_{t,k}(q)$ fixed points among the k-dimensional subspaces of $\mathbb{F}_q^n$.

3.4 Example: Computation of types

In this section we exhibit two examples: The first example shows how to determine the type of a matrix in general. The second example illustrates the computation of all types of matrices in $\mathrm{GL}(n,q)$ for a given n.

The computation of the type of a matrix

Here we follow [26] and we begin this part by introducing some notation. One can find more information on this notation in [25].

Definition 3.12: *Determinantal divisor*
Let A be an $n\times n$-matrix over a field $\mathbb{F}$ and let $A(x) = xI_n - A$. For k with $1 \le k \le n$ we define the k-th determinantal divisor $d_k(x)$ of A as

$$d_k(x) = \gcd\{\ \det(B)\ \mid\ B \textit{ is a } k\times k\textit{-submatrix of } A(x)\ \}. \tag{3.27}$$

We further define $d_0(x) = 1$.

The k-th determinantal divisor $d_k(x)$ is the greatest common divisor of all minors of degree k. These determinantal divisors form a divisibility chain

$$1 = d_0(x) \mid d_1(x) \mid \ldots \mid d_n(x) = \det(A(x)). \tag{3.28}$$

We define polynomials $q_k(x) = d_k(x)/d_{k-1}(x)$ for $1 \le k \le n$ and obtain the divisibility chain

$$q_1(x) \mid q_2(q) \mid \ldots \mid q_n(x). \tag{3.29}$$

The first, say $n-m$, polynomials are constant polynomials $q_i(x) = 1$ $(1 \le i \le n-m)$, but the remaining m polynomials are exactly the invariant factors of A. Therefore, we omit the first $n-m$ constant polynomials and then re-enumerate and rename the remaining polynomials. We obtain the invariant factors of A which are the m non-constant polynomials $a_1(x), \ldots, a_m(x)$. By decomposing every a_i into its monic irreducible factors we are able to read off the type of A.

Example 7. Let us consider the matrix A with

$$A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}. \tag{3.30}$$

We obtain

$$A(x) = \begin{pmatrix} x-2 & 2 & -14 \\ 0 & x-3 & 7 \\ 0 & 0 & x-2 \end{pmatrix} \qquad \text{and} \qquad \chi_A(x) = \det(A(x)) = (x-2)^2(x-3). \tag{3.31}$$

Now, we start the computation of the determinantal divisors.

1. By definition we have the factor $d_0(x) = 1$.
2. The first determinantal divisor is surely the greatest common divisor of all entries in $A(x)$. As we can choose all polynomials to be monic, we get the first determinantal divisor as $d_1(x) = 1$.
3. We get the second determinantal divisor $d_2(x)$ as the greatest common divisor of all determinants of 2×2-submatrices. Apart from the upper right block every such determinant is either zero or contains the factor $x - 2$. The top right 2×2-block has the determinant $14(x - 2)$. Thus, we find $d_2(x) = x - 2$.

4. The last determinantal divisor is exactly the determinant of $A(x)$, which is $(x-2)^2(x-3)$.

The determinantal divisors give us the invariant factors as their quotients. We obtain

$$a_1(x) = x - 2 \quad \text{and} \quad a_2(x) = (x-2)(x-3) = x^2 - 5x + 6. \tag{3.32}$$

Hence, the rational canonical form of A is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 1 & 5 \end{pmatrix}. \tag{3.33}$$

When splitting the invariant factors of A into their irreducible factors we obtain the elementary divisors. Let $p_1 = x - 2$ and $p_2 = x - 3$. Since p_1 occurs with multiplicity 1 in both a_1 and a_2, it is $s_1 = (1,1)$. Since p_2 only occurs once in a_2 it is $s_2 = (1)$. Hence, the type of A is

$$t = t_A = ((1,(1,1)),(1,(1))). \tag{3.34}$$

All types of matrices in $\mathrm{GL}(n,q)$ for a fixed n

One part of our `GAP` implementation [12] is the computation of all types of matrices in $\mathrm{GL}(n,q)$ for a given number $n \in \mathbb{N}$. The corresponding function is summarised in pseudo-code and can be found as Algorithm 4 in Appendix B (page 105).

The general idea of this algorithm is the following: First, we take all possible partitions of numbers $\leq n$. Let λ be such a partitions of length $l = l(\lambda)$. We identify every part λ_i of λ with the degree of an irreducible polynomial. Then, for every partition λ we need to find natural numbers $a_1, \ldots, a_l$ with $a_1\lambda_1 + \ldots + a_l\lambda_l = n$. The number a_i is the total multiplicity of the polynomial of degree λ_i. At last, the numbers a_i are replaced by partitions s_i of a_i. Then, (λ_i, s_i) is a type-parameter corresponding to an irreducible polynomial of degree λ_i. The total of all type-parameters is the type.

Example 8. Let $n = 3$. We determine all partitions of 1, 2, and 3 and obtain the sets $\{(1)\}$, $\{(2),(1,1)\}$, and $\{(3),(2,1),(1,1,1)\}$, respectively. So, we have to consider six partitions λ. Let us pick for example $\lambda = (1,1)$ and the only two possible solutions for (a_1,a_2) are $(2,1)$ and $(1,2)$. (When choosing $a_1 = 3$ we cannot find an $a_2 \in \mathbb{N}$ such that $3 \cdot 1 + a_2 \cdot 1 = n = 3$.) Now, we replace the numbers a_i by partitions of a_i. We start with $a = (1,2)$ and obtain the list of type-parameters

$$t_1 = ((1,(1)),(1,(2))) \qquad \text{and} \qquad t_2 = ((1,(1)),(1,(1,1))). \tag{3.35}$$

Doing the same for the element $(2,1)$ we analogously get

$$t_3 = ((1,(2)),(1,(1))) \qquad \text{and} \qquad t_4 = ((1,(1,1)),(1,(1))). \tag{3.36}$$

We sort the type-parameters lexicographically and get rid of duplicates. There remain the types t_3 and t_4. Possible representing matrices for the two types are given by

$$A_3 = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \qquad \text{and} \qquad A_4 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}. \tag{3.37}$$

Our algorithm appends some information to the type. We check if the representative of a type can be given as a block-diagonal-matrix. For those matrices holds that the first upper diagonal only contains of zero blocks. We do this because types represented by matrices having the same block structure along the main diagonal behave the same in many cases. For this purpose we fix the order of the elements $t \in T$ where T is the set of all types. Every t has a unique position within T. Therefore, in `GAP` we store the types as records with the following entries.

`.type` The type t itself. For this purpose we use lists containing the type-parameters (d, s) with d being the degree of an irreducible polynomial and s being the sequence of powers.

`.ss` This entry is set `true` if a representing matrix can be given as a block-diagonal-matrix. Otherwise we can find a type t_j with a representing matrix in block diagonal form such that the blocks along the main diagonal agree with the blocks of the given type. Then, the entry is set to `j`.

`.clen` This entry is a polynomial in q giving the number of elements within a conjugacy class of a matrix of the given type. It is the "**c**onjugacy class **len**gth".

Example 9. We complete here the list of all types of matrices in $\mathrm{GL}(3, q)$. In Table 3.2 it is $a, b, c \in \mathbb{F}_q \setminus \{0\}$ with $a \neq b \neq c \neq a$, it is $m_0, m_1 \in \mathbb{F}_q$ with $x^2 + m_1 x + m_0 \in \mathbb{F}_q[x]$ irreducible, and it is $n_0, n_1, n_2 \in \mathbb{F}_q[x]$ with $x^3 + n_2 x^2 + n_1 x + n_0$ irreducible.

Table 3.2: All possible types of matrices of $\mathrm{GL}(3, q)$.

number	.type	.ss	.clen	representative
1	`[ [[1,1,1],1] ]`	true	1	$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$
2	`[ [[2,1],1] ]`	1	$q^4 + q^3 - q - 1$	$\begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$
3	`[ [[3],1] ]`	1	$q^6 - q^4 - q^3 + q$	$\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$
4	`[ [[1,1],1], [[1],1] ]`	true	$q^4 + q^3 + q^2$	$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$
5	`[ [[2],1], [[1],1] ]`	4	$q^6 + q^5 - q^3 - q^2$	$\begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$
6	`[ [[1],1], [[1],1], [[1],1] ]`	true	$q^6 + 2q^5 + 2q^4 + q^3$	$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

Table 3.2: All possible types of matrices of $\mathrm{GL}(3,q)$.

number	.type	.ss	.clen	representative
7	`[ [[1],2], [[1],1] ]`	`true`	$q^6 - q^3,$	$\begin{pmatrix} 0 & -m_0 & 0 \\ 1 & -m_1 & 0 \\ 0 & 0 & a \end{pmatrix}$
8	`[ [[1],3] ]`	`true`	$q^6 - q^5 - q^4 + q^3$	$\begin{pmatrix} 0 & 0 & -n_0 \\ 1 & 0 & -n_1 \\ 0 & 1 & -n_2 \end{pmatrix}$

Chapter

4

Counting algebras

4.1 The covering algebra and descendants

Eick [9] developed an algorithm that generates all nilpotent associative algebras of dimension d over $\mathbb{F}_q$ up to isomorphism. This algorithm can be seen as an analogue to the p-group generation algorithm. We recall briefly Eick's algorithm as we base our algorithm on it. Throughout this section we follow the paper by Eick [9], we also refer to this paper for the proofs which are omitted in this section.

Let $\mathcal{A}$ be a nilpotent associative $\mathbb{F}_q$-algebra of class c and let $r = \dim(\mathcal{A}/\mathcal{A}^2)$ be its rank. Let $\mathcal{F}$ be the free associative $\mathbb{F}_q$-algebra without unit element with r generators. Every nilpotent associative $\mathbb{F}_q$-algebra of rank r must be isomorphic to a quotient of $\mathcal{F}$. Since $\mathcal{F}^{c+1}$ is generated by all products of length $c+1$ in $\mathcal{F}$ we get that $\mathcal{F}/\mathcal{F}^{c+1}$ is a free associative nilpotent $\mathbb{F}_q$-algebra of class c with r generators. Every class c nilpotent associative $\mathbb{F}_q$-algebra with r generators is isomorphic to a quotient of $\mathcal{F}/\mathcal{F}^{c+1}$.
The algebra $\mathcal{F}$ can be constructed in the following way (compare [30, Section 1.2]): We start with the polynomial algebra $\bar{\mathcal{F}}$ over $\mathbb{F}_q$ in r non-commuting indeterminates $x_1, \dots, x_r$. Then, we obtain $\mathcal{F}$ as its subalgebra of all polynomials with no constant term.
Now, let $\mathcal{R}$ be a two-sided ideal of $\mathcal{F}$ such that $\mathcal{F}/\mathcal{R} \cong \mathcal{A}$ for our given algebra $\mathcal{A}$. We define $\overline{\mathcal{R}}$ as the ideal generated by $\mathcal{F}\mathcal{R} \cup \mathcal{R}\mathcal{F}$. We further define $\mathcal{A}^* = \mathcal{F}/\overline{\mathcal{R}}$ and call it the **covering algebra** of $\mathcal{A}$. Additionally, we define $\mathcal{M} = \mathcal{R}/\overline{\mathcal{R}}$ and call it the **multiplicator** of $\mathcal{A}$.
By construction we get that $\mathcal{A}^*$ is a nilpotent associative $\mathbb{F}_q$-algebra with r generators of class c or $c+1$. Hence, $\mathcal{A}^*$ is a quotient of $\mathcal{F}/\mathcal{F}^{c+2}$ and it is finite dimensional. We further obtain $\mathcal{A} \cong \mathcal{A}^*/\mathcal{M}$ and so $\mathcal{A}^*$ is an extension of $\mathcal{A}$ by $\mathcal{M}$ where $\mathcal{M}$ is a module in which all products vanish. Such a module is called **zero-module**. With the notation as above we have $\mathcal{M} \leq (\mathcal{A}^*)^2$.

Theorem 4.1 ***(compare [9, Theorem 5])***
Let $\mathcal{A}$ be an associative nilpotent $\mathbb{F}_q$-algebra of class c and rank r. Every e generator extension of $\mathcal{A}$ by a zero-module is a quotient of $\mathcal{A}^*$. The isomorphism type of $\mathcal{A}^*$ depends only on $\mathcal{A}$ and is independent of the chosen ideal $\mathcal{R}$.

We define the **nucleus** of $\mathcal{A}^*$ as $\mathcal{N} = (\mathcal{A}^*)^{c+1}$. With the above notation we have $\mathcal{N} \leq \mathcal{M}$.
We can use the nucleus to define important subspaces of the multiplicator: Every proper subspace $U < \mathcal{M}$ which supplements the nucleus $\mathcal{N}$ to the whole multiplicator $\mathcal{M}$ is called an **allowable subspace** of $\mathcal{A}^*$. That is every $U < \mathcal{M}$ for which holds $U + \mathcal{N} = \mathcal{M}$. Note, that we just need the sum $U + \mathcal{N}$, we do not require this sum to be direct.

Example 10. Let us consider an r-generator $\mathbb{F}_q$-algebra $\mathcal{A}$ of class 1. As above we consider $\mathcal{F}$ to be the $\mathbb{F}_q$-algebra of polynomials without constant term in r non-commuting indeterminates $x_1, \ldots, x_r$. With $\mathcal{A}$ having class 1 we can choose $\mathcal{R} = \mathcal{F}^2$ and it follows $\mathcal{A} \cong \mathcal{F}/\mathcal{R}$. Considered as a vector space, $\mathcal{F}/\mathcal{R}$ is generated by $x_1, \ldots, x_r$, too.
Because the ideal $\bar{\mathcal{R}}$ is generated by $\mathcal{F}\mathcal{R} \cup \mathcal{R}\mathcal{F}$ we obtain in this situation that $\bar{\mathcal{R}}$ is equal to $\mathcal{F}^3$ and we have the covering algebra $\mathcal{A}^* \cong \mathcal{F}/\mathcal{F}^3$. As an $\mathbb{F}_q$-algebra $\mathcal{A}^*$ has still r generators, but it has class 2. Considered as a vector space, $\mathcal{A}^*$ is generated by $x_1, \ldots, x_r, x_1x_1, x_1x_2, \ldots, x_rx_r$ and, hence, its dimension as vector space over $\mathbb{F}_q$ is $r + r^2$.
The multiplicator is $\mathcal{M} = \mathcal{R}/\bar{\mathcal{R}} \cong \mathcal{F}^2/\mathcal{F}^3$. It has a trivial multiplication and we can handle $\mathcal{M}$ as a subspace of $\mathcal{A}^*$ of dimension r^2. A basis is given by the elements $x_1x_1, x_1x_2, \ldots, x_rx_r$.
We also observe that we obtain $\mathcal{A}^*$ as an extension of $\mathcal{A}$ by $\mathcal{M}$: As a vector space, a basis of $\mathcal{M}$ supplements a basis of $\mathcal{A}$ to a basis containing the generators of $\mathcal{A}$ and all products of two generators.
The nucleus is defined as $\mathcal{N} = (\mathcal{A}^*)^2$ and we get $\mathcal{N} \cong (\mathcal{F}/\mathcal{F}^3)^2 \cong (\mathcal{F}^2 + \mathcal{F}^3)/\mathcal{F}^3$. We can assume that $\mathcal{N}$ is generated by $x_1x_1, x_1x_2, \ldots, x_rx_r$.
In our case all products of elements of the nucleus are trivial. Hence, the nucleus is a zero-ideal. Considered as a vector space over $\mathcal{F}$, the nucleus is equal to the multiplicator and we find the allowable subspaces of $\mathcal{A}^*$: Because of $\mathcal{M} = \mathcal{N}$ every proper subspace of $\mathcal{M}$ supplements $\mathcal{N}$ to $\mathcal{M}$, so every proper subspace of $\mathcal{M}$ is allowable.

As in the case of the p-group generating algorithm (compare [28]) we now define descendants and determine the possible descendants of $\mathcal{A}^*$: A **descendant of** $\mathcal{A}$ is an associative $\mathbb{F}_q$-algebra $\mathcal{B}$ such that $\mathcal{B}$ is nilpotent of class $c+1$ and it is $\mathcal{A} \cong \mathcal{B}/\mathcal{B}^{c+1}$.

Theorem 4.2 ***(compare [9, Theorem 7])***

1. *Every descendant of $\mathcal{A}$ is a quotient $\mathcal{A}^*/U$ for some allowable subspace U of $\mathcal{A}^*$.*
2. *Every quotient $\mathcal{A}^*/U$ for an allowable subspace U of $\mathcal{A}^*$ is a descendant of $\mathcal{A}$.*

When trying to get all isomorphism types of descendants of $\mathcal{A}$ we need to check whether or not two quotients $\mathcal{A}^*/U$ and $\mathcal{A}^*/U'$ are isomorphic. Therefore, we consider the automorphism groups $\mathrm{Aut}(\mathcal{A}^*)$ and $\mathrm{Aut}(\mathcal{A})$. Let $\mathcal{B} \leq \mathcal{A}^2$ be an zero-ideal. That is an ideal such that all products of elements in $\mathcal{B}$ are trivial. We define

$$\mathrm{CAut}(\mathcal{A}, \mathcal{B}) = \left\{ \alpha \in \mathrm{Aut}(\mathcal{A}) \mid \alpha|_{\mathcal{A}/\mathcal{B}} = \mathrm{id}|_{\mathcal{A}/\mathcal{B}} \right\} \tag{4.1}$$

and call it the **central automorphisms of** $\mathcal{A}$ **in** $\mathcal{B}$. The following lemma will introduce some basic facts on these central automorphisms.

Lemma 4.3 ***(compare [9, Lemma 8])***
Let $\mathcal{A}$ be as above and let $\mathcal{B} \leq A^2$ an ideal such that all products in $\mathcal{B}$ are trivial. Then $\mathrm{CAut}(\mathcal{A}, \mathcal{B})$ is an elementary abelian p-group and it acts trivially on $\mathcal{B}$.

Equipped with the theory of descendants, the following two theorems allow us to decide whether or not two descendants of $\mathcal{A}$ are isomorphic.

Theorem 4.4 ***(compare [9, Theorem 9])***
We define $S = \{\alpha \in \mathrm{Aut}(\mathcal{A}^*) \mid \mathcal{M}^\alpha = \mathcal{M}\}$ and

$$\kappa : S \to \mathrm{Aut}(\mathcal{A}),\ \alpha \mapsto \alpha|_{\mathcal{A}^*/\mathcal{M}}. \tag{4.2}$$

1. *κ is surjective with kernel $\mathrm{CAut}(\mathcal{A}^*, \mathcal{M})$.*
2. *Let $\alpha \in \mathrm{Aut}(\mathcal{A})$ and let $\hat{\alpha}$ be an arbitrary preimage of α under κ. Let $\bar{\alpha} = \hat{\alpha}|_M \in \mathrm{Aut}(\mathcal{M})$ be its restriction on $\mathcal{M}$. Then $\bar{\alpha}$ does not depend on the choice of the mapping*

$$\varphi : \mathrm{Aut}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{M}),\ \alpha \mapsto \bar{\alpha}. \tag{4.3}$$

   Hence, φ is a well-defined homomorphism and thus yields an action of $\mathrm{Aut}(\mathcal{A})$ on $\mathcal{M}$.

Theorem 4.5 ***(compare [9, Theorem 10])***
Let $\mathcal{A}$ be as above and let X and Y be allowable subspaces with respect to $\mathcal{A}^*$.

1. *$\mathcal{A}^*/X \cong \mathcal{A}^*/Y$ if and only if there exists an $\alpha \in \mathrm{Aut}(\mathcal{A})$ with $X^{\bar{\alpha}} = Y$.*
2. *The natural map $\mathrm{Aut}(\mathcal{A}^*/X) \to \mathrm{Aut}(\mathcal{A}),\ \alpha \mapsto \alpha|_{\mathcal{A}^*/M}$, has the image $\mathrm{Stab}_{\mathrm{Aut}(\mathcal{A})}(X)$ and the kernel $\mathrm{CAut}(\mathcal{A}^*/X, M/X)$.*

4.2 Application of the covering algebra

In this section we will apply the theory of covering algebras (compare Section 4.1) to the case of nilpotent associative $\mathbb{F}_q$-algebras of class 1 to generate the nilpotent associative $\mathbb{F}_q$-algebras of class 2. There is only one nilpotent associative $\mathbb{F}_q$-algebra of class 1 up to isomorphism and, hence, the algorithm needs to be applied to just one class 1 $\mathbb{F}_q$-algebra. In Example 10 we already considered this class 1 $\mathbb{F}_q$-algebra.
We use the same notation as in Section 4.1 and specify the algebra generation algorithm by Eick (see [9]) to our case to generate the nilpotent associative $\mathbb{F}_q$-algebras of class 2.

Let $T(r, \mathbb{F}) = \mathbb{F}^r \otimes_{\mathbb{F}} \mathbb{F}^r$. When working with the finite field with q elements we will abbreviate the tensor product space by $T(r, q)$. Further, we define $\mathcal{U}_k(r, \mathbb{F})$ as the set of k-dimensional subspaces of $T(r, \mathbb{F})$. Again, we will abbreviate this notation by $\mathcal{U}_k(r, q)$ when the field is $\mathbb{F}_q$. Now, Theorem 4.6 states how we are able to compute the number $N_{d,r}(q)$ of isomorphism types of nilpotent associative $\mathbb{F}_q$-algebras of class 2 and dimension d.

Theorem 4.6 *The number $N_{r+k,r}(q)$ coincides with the number of orbits of $\mathrm{GL}(r,q)$ on $\mathcal{U}_k(r,q)$.*

Proof: We start with a nilpotent associative $\mathbb{F}_q$-algebra of class 1 with r generators which is unique up to isomorphism. A presentation of $\mathcal{A}$ is given by

$$\mathcal{A} \cong \langle a_1, \ldots, a_r \mid a_i a_j = 0 \text{ for } 1 \leq i,j \leq r \rangle . \tag{4.4}$$

We apply the theory of descendants to $\mathcal{A}$ (compare Section 4.1). Let $\mathcal{F}$ denote the free associative $\mathbb{F}_q$-algebra on r generators without unit element.
Using the ideal $\mathcal{R} = \mathcal{F}^2 \trianglelefteq \mathcal{F}$ we get that $\mathcal{A} \cong \mathcal{F}/\mathcal{R} = \mathcal{F}/\mathcal{F}^2$ is a nilpotent associative $\mathbb{F}_q$-algebra with r generators of class 1. So, $\mathcal{A}$ can be given by the presentation in Equation (4.4). Its dimension is $\dim_{\mathbb{F}_q}(\mathcal{A}) = r$.
The covering algebra of $\mathcal{A}$ is $\mathcal{A}^* = \mathcal{F}/\mathcal{F}^3$ since we have $\bar{\mathcal{R}} = \langle \mathcal{F}\mathcal{R} \cup \mathcal{R}\mathcal{F} \rangle = \mathcal{F}^3$. The covering algebra has class 2 and we obtain the power ideal series $\mathcal{A}^* > (\mathcal{A}^*)^2 > (\mathcal{A}^*)^3 = \{0\}$. This series yields quotients whose dimensions are given by

$$\dim_{\mathbb{F}_q}(\mathcal{A}^*/(\mathcal{A}^*)^2) = \dim_{\mathbb{F}_q}(\mathcal{F}/\mathcal{F}^2) = r \qquad \text{and} \qquad \dim_{\mathbb{F}_q}((\mathcal{A}^*)^2/(\mathcal{A}^*)^3) = r^2. \tag{4.5}$$

A presentation of the covering algebra is therefore given by

$$\mathcal{A}^* \cong \langle a_1, \ldots, a_r, b_{11}, b_{12}, \ldots, b_{rr} \mid a_i a_j = b_{ij}, a_i b_{jk} = b_{jk} a_i = 0 \text{ for } 1 \leq i,j,k \leq r \rangle . \tag{4.6}$$

We recognise: $\mathcal{M} = \mathcal{R}/\bar{\mathcal{R}} = \mathcal{F}^2/\mathcal{F}^3 \cong ((\mathcal{F}^2/\mathcal{F}^3)/(\mathcal{F}^3/\mathcal{F}^3)) = (\mathcal{A}^*)^2/(\mathcal{A}^*)^3$. Hence, a presentation for the multiplicator is given by

$$\mathcal{M} \cong \langle b_{11}, b_{12}, \ldots, b_{rr} \mid b_{ij} b_{i'j'} = 0 \text{ for } 1 \leq i,j,i',j' \leq r \rangle . \tag{4.7}$$

For the nucleus it holds that $\mathcal{N} = (\mathcal{A}^*)^{\mathrm{cl}(\mathcal{A})+1} = (\mathcal{A}^*)^2 \cong \mathcal{F}^2/\mathcal{F}^3 \cong \mathcal{M}$. So, the set of allowable subspaces $U \leq \mathcal{M}$ is given by the set of all proper subspaces of $\mathcal{M}$. As a vector space we can identify $\mathcal{M}$ with the tensor product space $T(r,q) = \mathbb{F}_q^r \otimes_{\mathbb{F}_q} \mathbb{F}_q^r$. The set of allowable subspaces is given by

$$\{U \leq T(r,q) \mid U \text{ is allowable subspace}\} = \bigcup_{k=0}^{r^2-1} \mathcal{U}_k(r,q). \tag{4.8}$$

Now, we want to apply Theorem 4.4 to this situation. We begin by determining $\mathrm{Aut}(\mathcal{A})$: As $\mathcal{A}$ is an $\mathbb{F}_q$-algebra with r generators which has trivial multiplication every vector space isomorphism is compatible with the multiplication, hence, it is $\mathrm{Aut}(\mathcal{A}) \cong \mathrm{GL}(r,q)$.
Next, we determine $\mathrm{Aut}(\mathcal{A}^*)$. Consider the epimorphism κ as in Theorem 4.4 and, analogously to the theorem, let $S = \{\alpha \in \mathrm{Aut}(\mathcal{A}^*) \mid \mathcal{M}^\alpha = \mathcal{M}\}$. With respect to the chosen presentation of $\mathcal{A}^*$ (see Equation (4.6)) we get

$$\mathrm{Aut}(\mathcal{A}^*) \cong \{g \oplus (g \otimes g) \mid g \in \mathrm{GL}(r,q)\} \leq \mathrm{GL}(r + r^2, q). \tag{4.9}$$

Hence, any $\alpha \in \mathrm{Aut}(\mathcal{A})$ can be represented by some matrix $g \in \mathrm{GL}(r,q)$ and its preimage under κ is given by $\hat{\alpha}$ represented by $g \oplus (g \otimes g)$. When restricting $\hat{\alpha}$ to $\mathcal{M}$ it follows that $\bar{\alpha}$ is represented by $g \otimes g$.
We apply Theorem 4.5 which states that two descendants $\mathcal{A}^*/X$ and $\mathcal{A}^*/Y$ with proper subspaces $X, Y < \mathcal{M} = T(r,q)$ are isomorphic if and only if there exists an $\bar{\alpha} \in \mathrm{Aut}(\mathcal{M})$ such that $X^{\bar{\alpha}} = Y$. We have already shown that $\bar{\alpha}$ can be represented by some matrix $g \otimes g$ with $g \in \mathrm{GL}(r,q)$.
Let now $\mathcal{B}$ be an associative nilpotent $\mathbb{F}_q$-algebra with r generators of class 2 having dimension $r+k$. Applying Theorem 4.2 we find a proper subspace $X < T(r,q)$ such that $\mathcal{B} \cong \mathcal{A}^*/X$ with $\dim_{\mathbb{F}_q}(X) = r^2 - k$. (Recall that $\dim_{\mathbb{F}_q}(\mathcal{A}^*) = r + r^2$.)

The number of isomorphism types of $r+k$-dimensional nilpotent associative $\mathbb{F}_q$-algebras of rank r and class 2 agrees with the number of orbits of $\mathrm{GL}(r,q)$ on $\mathcal{U}_{r^2-k}(r,q)$. There is a bijection $\mathcal{U}_{r^2-k}(r,q) \to \mathcal{U}_k(r,q)$ sending a vector space in $\mathcal{U}_{r^2-k}(r,q)$ to its orthogonal complement with respect to the standard basis and the standard scalar product. Because of this duality we have: The number $N_{r+k,r}(q)$ can be given by the number of orbits of $\mathrm{GL}(r,q)$ on $\mathcal{U}_k(r,q)$. $\square$

We can use the result of Theorem 4.6 to develop an algorithm for the computation of $N_{d,r}(q)$. We combine this result with the counting lemma by Burnside, Cauchy and Frobenius. A proof of this lemma can for example be found in [31, Theorem 3.22].

Lemma 4.7 ***(Lemma of Burnside, Cauchy and Frobenius)***
Let X be a set and let G be a finite group acting on X. For $g \in G$ let $X^g := \{x \in X \mid gx = x\}$ be the set of elements fixed by g. The number of orbits $|X/G|$ is then given by

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|. \tag{4.10}$$

Combining Theorem 4.6 and Lemma 4.7 leads to a formula for $N_{r+k,r}(q)$.

Corollary 4.8 *Let $\mathrm{Fix}_g(\mathcal{U}_k(r,q))$ denote the number of fixed points of $\mathcal{U}_k(r,q)$ under the action of $g \in \mathrm{GL}(r,q)$. Then,*

$$N_{r+k,r}(q) = \frac{1}{|\mathrm{GL}(r,q)|} \sum_{g \in \mathrm{GL}(r,q)} \mathrm{Fix}_g(\mathcal{U}_k(r,q)). \tag{4.11}$$

We notice that in Corollary 4.8 the sum is applied to all elements of the group $\mathrm{GL}(r,q)$. However, we want to keep q as a variable and we want to obtain $N_{r+k,r}(q)$ as a function in q. We need to improve the formula in Corollary 4.8 to get the summation independent of q.
We have already seen (compare Lemma 3.11) that the number of types of matrices in $\mathrm{GL}(r,q)$ does not depend on q. Therefore, we will reformulate Corollary 4.8 such that we are just having a sum over the types of matrices.
Let now $g \in \mathrm{GL}(r,q)$ act on $\mathrm{Fix}_g(\mathcal{U}_k(r,q))$ and let g be of type $t = t(g)$. We have the set $\mathcal{O}(t) = \{g \otimes g \mid g \in \mathcal{E}(t)\}$ where $\mathcal{E}(t)$ denotes the set of all elements of type t. Furthermore, let $\mathcal{T}$ be the set of all types of $\mathrm{GL}(r,q)$. Instead of having a sum over all elements themselves we iterate over the set of all types $t \in \mathcal{T}$. Then, for every type t we run over the set of all possible types of matrices in $\mathcal{O}(t)$.
For that reason, let $\mathcal{R}(t)$ be a set of representatives for the conjugacy classes of all matrices of type t. We define $A(t,\bar{t})$ as the number of representatives in $\mathcal{R}(t)$ which yield type $\bar{t}$ after the application of the Kronecker product. Hence, it is

$$A(t,\bar{t}) = \{g \in \mathcal{R}(t) \mid g \otimes g \text{ has type } \bar{t}\}. \tag{4.12}$$

We denote the size of the conjugacy classes of elements of type t by $|cc(t)|$. At last we write $F_k(\bar{t})$ for the number of fixed points under the action of an element of type $\bar{t}$ on the set $\mathcal{U}_k(r,q)$. This number only depends on the type and not on a chosen representing matrix, see Lemma 3.11.
With the above notation we can bring Equation (4.11) into the following form:

$$N_{r+k,r}(q) = \frac{1}{|\mathrm{GL}(r,q)|} \sum_{t \in \mathcal{T}} \sum_{\bar{t} \in \mathcal{O}(t)} |cc(t)| \cdot A(t,\bar{t}) \cdot F_k(\bar{t}). \tag{4.13}$$

Note, that $\varphi : \mathrm{GL}(r,q) \to \mathrm{GL}(r^2,q)$, $g \mapsto g \otimes g$, is a homomorphism of groups. Thus, two matrices of type $\bar{t}$ are conjugate if their underlying matrices of type t are already conjugate. Because of this we can replace $|cc(\bar{t})|$ in Equation (4.13) by $|cc(t)|$. It is independent of the summation index $\bar{t}$ and we can write it in front of the second sum. We now obtain the coefficient $\frac{|cc(t)|}{|\mathrm{GL}(r,q)|}$ which is equal to $\frac{1}{c_t}$, see Lemma 3.9 and Lemma 3.10. It follows:

$$N_{r+k,r}(q) = \sum_{t \in \mathcal{T}} \frac{1}{|\mathrm{GL}(r,q)|} \cdot |cc(t)| \sum_{\bar{t} \in \mathcal{O}(t)} A(t,\bar{t}) \cdot F_k(\bar{t}) = \sum_{t \in \mathcal{T}} \frac{1}{c_t} \sum_{\bar{t} \in \mathcal{O}(t)} A(t,\bar{t}) \cdot F_k(\bar{t}). \tag{4.14}$$

In summary, we conclude the following corollary which will be formed into our algorithm.

Corollary 4.9 *Let $\mathrm{GL}(r,q)$ act on the set $\mathcal{U}_k(r,q)$ of k-dimensional subspaces of $T(r,q)$ and let $\mathcal{T}$ be the set of all types of matrices in $\mathrm{GL}(r,q)$. Let $g \in \mathrm{GL}(r,q)$ act on $\mathcal{U}_k(r,q)$ via the matrix $g \otimes g$ of type $\bar{t} \in \mathcal{O}(t)$. Let $F_k(\bar{t})$ be the number of fixed points under g acting on $\mathcal{U}_k(r,q)$. Then,*

$$N_{r+k,r}(q) = \sum_{t \in T} \frac{1}{c_t} \sum_{\bar{t} \in \mathcal{O}(t)} A(t,\bar{t}) \cdot F_k(\bar{t}). \tag{4.15}$$

We have now reformulated Equation (4.13) in a way that we do not sum over every group element itself, but only over their types. We use Corollary 4.9 to formulate our algorithm (see Section 4.3).
Furthermore, we can use the results of this section to proof our two main results Theorem 1.3 and 1.4. Recall that Theorem 1.3 states that the number $N_{r+k,r}(q)$ is PORC.

Proof (Theorem 1.3):
By Lemma 3.11 we obtain that $F_k(\bar{t})$ is a polynomial in q. Lemma 3.10 states that c_t can be considered as a polynomial in q. The constructions of Equation (4.15) yields that c_t divides $\sum_{\bar{t} \in \mathcal{O}(t)} A(t,\bar{t}) F_k(\bar{t})$. Corollary 5.12 in Section 5.2 will show that $A(t,\bar{t})$, considered as a function in q, is PORC for all types t and $\bar{t}$. Hence, the number $N_{r+k,r}(q)$, considered as a function in q, is PORC. □

We can also conclude the correctness of Theorem 1.4 which established the following properties of the functions $N_{d,r}(q)$:

1. $N_{d,r}(q) = 0$ if $d \notin \{r+1, \ldots, r+r^2\}$,
2. $N_{r+r^2,r}(q) = 1$, and
3. $N_{r+k,r}(q) = N_{r+r^2-k,r}(q)$ for all $k \in \{1, \ldots, r^2-1\}$.

Proof (Theorem 1.4):
The first part follows because $\mathcal{U}_k(r,q)$ is the empty set if k is a negative number or if $k > r^2$. In those cases we cannot obtain any orbits. We developed the computation of $N_{d,r}(q)$ using the theory of descendants. There, allowable subspaces were defined as non-trivial subspaces or the zero-subspace. That is why there is no function $N_{d,r}(q)$ for $d = r$. Then, it follows $N_{d,r}(q) = N_{r+k,r}(q) = 0$ if $d \notin \{r+1, \ldots, r+r^2\}$.
For the second part we note that there is just one subspace of dimension 0 and, hence, we obtain $N_{r+r^2,r}(q) = 1$.
The last part follows because of the duality between k-dimensional and r^2-k-dimensional subspaces of $T(r,q)$. Choosing the canonical basis for the vector space $T(r,q)$ and the canonical scalar multiplication, for every k-dimensional subspace U there is a unique r^2-k-dimensional complement. The canonical scalar product is compatible with the Kronecker product. We get a bijection $\mathcal{U}_k(r,q) \to \mathcal{U}_{r^2-k}(r,q)$ and so we obtain the duality $N_{r+k,r^2}(q) = N_{r+r^2-k,r}(q)$. □

4.3 The algorithm

In Section 4.1 we introduced the theory for our algorithm, in Section 4.2 we formulated the necessary theorems for our implementation (see especially Corollary 4.9). Here, we give a brief summary of our main algorithm in pseudo-code (compare Algorithm 1).
For a fixed rank r we first have to find the set $\mathcal{T}$ all types t. That was already introduced in Section 3.4 and we can use Algorithm 4 (see Appendix B) for this purpose.
Then, we must determine the set $\mathcal{O}(t) = \{g \otimes g \mid g \in \mathcal{E}(t)\}$ and the numbers $A(t, \bar{t})$. We do these two things together and this is the most time-consuming step of the algorithm. We will introduce this algorithm in detail in Chapter 5.
At last we need to determine the fixed points of the action. For this purpose we use the algorithm `FixedPointsByDecom` which is contained in the package `AutPGrp` (by Eick and O'Brien [11]) of the computer algebra system `GAP` [15].
We obtain our algorithm for the computation of $N_{d,r}(q)$: The algorithm `NumberOfClassTwoAlgebras` (Algorithm 1) has a parameter `act`: The `Kronecker` action determines for a matrix of type t all possible types $\bar{t}$ of matrices $g \otimes g$ as introduced within this section. It is possible to extend this algorithm by further actions. Then it is possible to compute enumerating functions for different algebraic objects.

Algorithm 1: `NumberOfClassTwoAlgebras(` **r**, **act**[, **k**])
The complete algorithm – determination of PORC functions for a fixed rank **r** and all dimensions **d** depending on the given action **act**. For instance, **act**=`Kronecker` applies the Kronecker product and yields the number of isomorphism types of nilpotent associative algebras of class two.
If an optional argument **k** is given, only PORC functions for dimension **k**+**r** is computed.

Input: Natural number **r**;
Output: PORC functions for all possible dimensions (depending on **r**);

Determine the set of possible types **T** of Elements of $\mathrm{GL}(r, q)$; # *See Algorithm 4*

for `each type t in T` **do**
  Determine the types $\bar{\mathbf{t}}$ of elements in $\mathcal{O}(\mathbf{t}) = \{g \otimes g \mid g \in \mathcal{E}(\mathbf{t})\}$; # *See Algorithm 7*
  Determine the numbers **A(t, $\bar{\mathbf{t}}$)**; # *See Algorithm 7*
  for `1` $\leq$ `k` $\leq$ $\lceil$`(r^2-1)/2`$\rceil$ **do**
    for `each` $\bar{\mathbf{t}}$ **do**
      determine **F_k($\bar{\mathbf{t}}$)**; # *Use* `FixedPointsByDecom` *from the* `AutPGrp` *package, see [11]*
    od;
  od;
od;

for k `in` $\{1, \ldots, \lceil (r^2-1)/2 \rceil\}$ **do**
  Determine $f_k := \sum_{t \in T} \frac{1}{c_t} \sum_{\bar{t} \in O(t)} \mathcal{A}(t, \bar{t}) \cdot F_k(\bar{t})$;
od;

return $\{f_1, \ldots, f_l\}$ `with` $l = \lceil (r^2-1)/2 \rceil$;

Chapter 5

Counting elements by types

5.1 Application of the Kronecker product

In Section 3.2 the term *type of a matrix* was introduced and in Section 3.4 an algorithm was introduced that computes all types of matrices in $\mathrm{GL}(r,q)$ for a given r. In Chapter 4 we formulated an algorithm to determine $N_{d,r}(q)$. However, it remains to develop an algorithm which determines for a given type t all types $\bar{t}$ of matrices in $\mathcal{O}(t) = \{g \otimes g \mid g \in \mathcal{E}(t)\}$. Additionally, this algorithm shall compute for each $\bar{t}$ the number $A(t,\bar{t})$. Recall that $A(t,\bar{t})$ is the number of distinct conjugacy classes of matrices of type $\bar{t}$ in $\mathcal{O}(t)$.
Throughout this chapter let t be a type of matrices in $\mathrm{GL}(n,q)$ with

$$t = ((n_1, s_1), \ldots, (n_m, s_m)) \quad \text{and} \quad s_i = (s_{i,1}, \ldots, s_{i,m_i}). \tag{5.1}$$

For every type-parameter (n_i, s_i) we define $\mathbb{E}_i$ to be the field extension of $\mathbb{F}_q$ of degree n_i. Let $\hat{\mathbb{E}}_i \subseteq \mathbb{E}_i$ be the set of all elements $x \in \mathbb{E}_i$ which are not contained in any proper subfield of $\mathbb{E}_i$. Further, let $\mathbb{E}$ be the smallest field extension of $\mathbb{F}_q$ which contains all fields $\mathbb{E}_i$, hence, $\mathbb{E}$ is a field extension of $\mathbb{F}_q$ of degree $\mathrm{lcm}(n_1, \ldots, n_m)$.
For a fixed type t we know that every matrix $g \in \mathrm{GL}(r,q)$ of type t is similar to a matrix in Jordan normal form over the field $\mathbb{E}$.

Definition 5.1: ***Jordan block***
Let $a \in \mathbb{E} \setminus \{0\}$ and let $r \in \mathbb{N}$. The Jordan block of size r and eigenvalue a, denoted by $J_r(a)$, is the $r \times r$-matrix given as

$$J_r(a) = \begin{pmatrix} a & a & 0 & & 0 \\ 0 & a & a & \ddots & \\ & \ddots & \ddots & \ddots & 0 \\ 0 & & 0 & a & a \\ 0 & 0 & & 0 & a \end{pmatrix} \in \mathbb{E}^{r \times r}. \tag{5.2}$$

When writing a Jordan block there are usually 1's along the first upper diagonal. However, Jordan blocks in both notations are conjugate to each other over $\mathrm{GL}(\mathbb{E}, r)$. Using the matrix

$$M = \begin{pmatrix} a & a & a & \dots & a & a \\ 0 & a^2 & a^2 & & a^2 & a^2 \\ 0 & 0 & a^3 & & a^3 & a^3 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & a^{r-1} & a^{r-1} \\ 0 & 0 & 0 & \cdots & 0 & a^r \end{pmatrix} \in \mathrm{GL}(r, \mathbb{E})$$

one obtains that $MJ_r(a)M^{-1}$ is a Jordan block of length r with 1's along the first upper diagonal.

Theorem 5.2 *Let $g \in \mathrm{GL}(r, q)$ be an element of type $t = ((n_1, s_1), \dots, (n_m, s_m))$. Then, there exist elements $a_1, \dots, a_m$ with $a_i \in \hat{\mathbb{E}}_i$ for $1 \leq i \leq m$ such that the elements in*

$$\left\{ a_i^{q^k} \mid 1 \leq i \leq m,\, 0 \leq k < n_i \right\} \tag{5.3}$$

are pairwise different. Further, we find a matrix $\hat{g}$ which is conjugate to g over $\mathrm{GL}(r, \mathbb{E})$ and which has the form

$$\hat{g} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m_i} \bigoplus_{k=0}^{n_i - 1} J_{s_{i,j}} \left(a_i^{q^k} \right). \tag{5.4}$$

The element $\hat{g}$ is unique up to permutation of Jordan blocks.

Proof: Due to the definition of types we can find a matrix $\bar{g} \in \mathrm{GL}(n, q)$ in rational canonical form such that g is conjugate to $\bar{g}$ (over $\mathbb{F}_q$). Thus, $\bar{g}$ is a direct sum of companion matrices corresponding to polynomials $p_i(x)^{s_{i,j}}$ where $p_i(x) \in \mathbb{F}_q[x]$ is irreducible of degree n_i.
Now, $\mathbb{E}_i$ is a splitting field for $p_i(x)$ and we can find a root $a_i \in \hat{\mathbb{E}}_i$ of $p_i(x)$ such that

$$p_i(x) = (x - a_i)(x - a_i^q) \cdot \ldots \cdot (x - a_i^{q^{n_i - 1}}) \in \mathbb{E}_i[x]. \tag{5.5}$$

Over $\mathbb{E}_i$ (and also over $\mathbb{E}$) the companion matrix of $p_i(x)^{s_{i,j}}$ is conjugate to a direct sum of Jordan blocks $J_{s_{i,j}}(a_i^{q^k})$ with $k \in \{0, \dots, n_i - 1\}$.
As all polynomials $p_i(x)$ are pairwise different, the roots $a_i \in \hat{\mathbb{E}}_i$ are pairwise different, too. Now, the result of the theorem follows immediately. □

Using Theorem 5.2 we are able to derive an alternative formula for the number k_t which is the number of different conjugacy classes of matrices in $\mathcal{E}(t)$. We recall that $\pi(t)$ is the number of permutations of the list of type-parameters $((n_1, s_1), \dots, (n_m, s_m))$ which do not change the type t (compare the proof of Lemma 3.10).

Corollary 5.3 *For $i \leq i \leq m$ define $c_i = |\{j \in \{1, \dots, i-1\} \mid n_j = n_i\}$. Then, it is*

$$k_t = \frac{1}{\pi(t) \cdot n_1 \cdot \ldots \cdot n_m} \prod_{i=1}^{m} \left(\left| \hat{\mathbb{E}}_{n_i} \right| - c_i \cdot n_i \right). \tag{5.6}$$

Proof: We start with the number of polynomials of degree n_k which are irreducible over the field $\mathbb{F}_q$. By convention, the number $|\hat{\mathbb{E}}_{n_k}|$ gives the number of possible roots of those polynomials considered over their splitting fields. Every such polynomial has got n_k different roots and, furthermore, the roots of two distinct irreducible polynomials are distinct, too. So, the number of degree n_k polynomials being irreducible over $\mathbb{F}_q$ is the same as $|\hat{\mathbb{E}}_k|/n_k$. Thus, we get that

$$\prod_{i=1}^{m}(|\hat{\mathbb{E}}_{n_i}|/n_i - c_i) \tag{5.7}$$

is the number of distinct irreducible polynomials over $\mathbb{F}_q$ that can be chosen for a representing matrix of type t. The number c_i ensures that the polynomials are distinct. As the ordering of the blocks does not change the type, we have to cancel out the number of possible permutations. Hence, we get the formula stated in the corollary. □

Next, we analyse the possible types of elements in $\mathcal{O}(t)$. We have seen (compare Theorem 5.2) that for every type parameter (n_i, s_i) we can find a matrix

$$g_i := \bigoplus_{j=1}^{m_i} \bigoplus_{k=0}^{n_i-1} J_{s_{i,j}}\left(a_i^{q^k}\right) \tag{5.8}$$

such that $a_i \in \hat{\mathbb{E}}_i$ and such that

$$\prod_{k=0}^{n_i-1}(x - a_i^{q^k}) \tag{5.9}$$

is an irreducible polynomial in $\mathbb{F}_q[x]$ of degree n_i. Hence, we can represent a matrix g of a certain type t as a matrix of the form

$$g := \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m_i} \bigoplus_{k=0}^{n_i-1} J_{s_{i,j}}\left(a_i^{q^k}\right). \tag{5.10}$$

The construction of Kronecker products is compatible with the block structure of matrices and the following lemma is a conclusion of Theorem 5.2.

Lemma 5.4 *Let $g \in \mathrm{GL}(r, q)$ be a matrix of type t $= ((n_1, s_1), \ldots, (n_m, s_m))$. Then, $g \otimes g$ is conjugate in $\mathrm{GL}(r^2, \mathbb{E})$ to*

$$\hat{g} \otimes \hat{g} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m_i} \bigoplus_{k=0}^{n_i-1} \bigoplus_{i'=1}^{m} \bigoplus_{j'=1}^{m_{i'}} \bigoplus_{k'=0}^{n_{i'}-1} J_{s_{i,j}}\left(a_i^{q^k}\right) \otimes J_{s_{i',j'}}\left(a_{i'}^{q^{k'}}\right). \tag{5.11}$$

Proof: This follows immediately from Theorem 5.2 together with the definition of the Kronecker product. □

To shorten notation we agree on the following: Given a type t and a matrix $g \in \mathrm{GL}(r, \mathbb{E})$ of type t in Jordan normal form, then $\bar{g} \in \mathrm{GL}(r^2, \mathbb{E})$ is a matrix in Jordan normal form conjugate to $g \otimes g$ having type $\bar{t}$. We especially have $\bar{g} \in \mathcal{O}(t)$. Note, that $\bar{g}$ (and therefore $\bar{t}$, too) strongly depends on the chosen elements $a_i \in \mathbb{E}_i$.
To obtain all possible types of a matrix $\bar{g}$ we have to answer the following questions:

1. How does a Kronecker product $J_r(a) \otimes J_r(b)$ decompose into Jordan blocks?

2. Having found such a decomposition, are there any coincidences among the different Jordan blocks?

To answer the first question we will consider two Jordan blocks $J_r(a) \in \mathrm{GL}(r, \mathbb{E})$ and $J_s(b) \in \mathrm{GL}(s, \mathbb{E})$ with $a, b \in \mathbb{E}$. We are interested in the Jordan normal form of $J_r(a) \otimes J_s(b)$ and for this purpose we write

$$A_{r,s} = J_r(1) \otimes J_s(1) \in \mathrm{GL}(rs, \mathbb{Z}). \tag{5.12}$$

For computational reasons it is useful to handle $A_{r,s}$ as an integer matrix instead of a matrix in $\mathrm{GL}(rs, \mathbb{E})$. This interpretation works by sending $0 \in \mathbb{E}$ to $0 \in \mathbb{Z}$ and analogously $1 \in \mathbb{E}$ to $1 \in \mathbb{Z}$ because $A_{r,s}$ contains only 1's and 0's.
We further define I_{rs} to be the identity matrix in $\mathrm{GL}(rs, \mathbb{Z})$. Now, we compute for $0 \leq i \leq rs + 1$ the matrix $(A_{r,s} - I_{rs})^i$ and denote its elementary divisors with $(b_{i,1}, \ldots, b_{i,rs})$. We further define numbers $a_i(p) = |\{j \in \{1, \ldots, rs\} \text{ with } p \mid b_{i,j}\}|$. Here p is a prime number. Then let

$$d_i(p) = 2a_i(p) - a_{i-1}(p) - a_{i+1}(p). \tag{5.13}$$

The values $d_i(p)$ give the number of Jordan blocks of size i within the Jordan normal form of $A_{r,s}$. This decomposition depends on the characteristic of the field $\mathbb{F}_q$ which is given by p.

Theorem 5.5 *Given $r, s \in \mathbb{N}$ and $a, b \in \mathbb{E}$ with $p = \mathrm{char}(\mathbb{E})$. Then $J_r(a) \otimes J_s(b)$ is conjugate in $\mathrm{GL}(rs, \mathbb{E})$ to*

$$\bigoplus_{i=1}^{rs} \bigoplus_{j=1}^{d_i(p)} J_i(ab). \tag{5.14}$$

Proof: We have $J_r(a) \otimes J_s(b) = abA_{r,s}$. So, it is sufficient to consider the matrix $A_{r,s}$. We consider $A_{r,s}$ as a matrix in $\mathrm{GL}(rs, \mathbb{E})$. The number $a_i(p)$ is the dimension of the kernel of $(A_{r,s} - I_{rs})^i$ considered as a matrix over the field $\mathbb{E}$ with characteristic p. By construction, the number $d_i(p)$ is the number of Jordan blocks $J_i(ab)$ of the matrix $abA_{r,s}$ having exactly length i. □

As Theorem 5.5 shows, the Jordan decomposition of $A_{r,s}$ is characterised by the values $d_i(p)$ which depend on the characteristic p of the field $\mathbb{F}_q$. Therefore, let

$$b = \max\{\, b_{i,j} \mid 0 \leq i \leq rs + 1,\ 1 \leq j \leq rs\}. \tag{5.15}$$

Hence, it is $a_i(p) = |\{j \in \{1, \ldots, rs\} \mid b_{i,j} = 0\}|$ for $p > b$ and thus, for all primes $p > b$ the $d_i(p)$ are independent of p. For all fields $\mathbb{E}$ with characteristic $\mathrm{char}(\mathbb{E}) > b$ the Jordan normal form is independent of $\mathbb{E}$. We call the primes $p \leq b$ **exceptional primes**.
For the primes that are not exceptional we can use the following Theorem 5.6 which originally determines the Jordan normal form of $J_r(1) \otimes J_s(1)$ over the field of complex numbers. Note, that its constructive proof directly translates to finite fields of characteristic p provided that p is not exceptional. Hence, Theorem 5.6 yields the normal form of $J_r(a) \otimes J_s(b) = ab(J_r(1) \otimes J_s(1))$ if $\mathrm{char}(\mathbb{E})$ is not exceptional and thus, for all characteristics $p > b$.

Theorem 5.6 ***(compare [26, Chapter 7, Theorem 1.2 and 1.3])***
Let $r, s \in \mathbb{N}$ and $a, b \in \mathbb{C}$. Then $J_r(1) \otimes J_s(1)$ is conjugate in $\mathrm{GL}(rs, \mathbb{C})$ to the direct sum of matrices $J_{d_1}(1) \oplus \ldots \oplus J_{d_l}(1)$, where $l = \min\{r, s\}$ and $d_i = r + s + 1 - 2i$ for $1 \leq i \leq l$.

Example 11. Let us take the Jordan blocks of length $r = 3$ and $s = 2$ and let us compute $J_r(1) \otimes J_s(1)$. Following the proof of Theorem 5.5 we consider the matrix

$$A_{3,2} - I_6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - I_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.16}$$

We apply the method for computing the elementary divisors which has been introduced in Section 3.4. Note, that this time the computation is done over the ring $\mathbb{Z}$ instead of $\mathbb{F}_q[x]$. All minors of size ≤ 6 have to be computed. The results are summarised in Table 5.1.

Power i of $(A_{3,2} - I_6)^i$	Elementary divisors	$a_i(2)$	$a_i(3)$	$a_i(p)$, $p \geq 5$
$i = 0$	$1 \mid 1 \mid 1 \mid 1 \mid 1 \mid 1$	0	0	0
$i = 1$	$1 \mid 1 \mid 1 \mid 1 \mid 0 \mid 0$	2	2	2
$i = 2$	$1 \mid 1 \mid 0 \mid 0 \mid 0 \mid 0$	4	4	4
$i = 3$	$3 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0$	5	6	5
$i \geq 4$	$0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0$	6	6	6

Table 5.1: Elementary divisor chains of the powers of the matrix from Equation (5.16). As the power $i = 4$ already yields the zero matrix, the divisibility chain will not change when increasing the power. The values $a_i(p)$ are also 6 for all higher powers.

With Table 5.1 we are able to compute the values $d_i(p)$. It follows:

$$d_1(p) = 0, \quad d_2(p) = 1, \quad d_3(p) = 0, \quad d_4(p) = 1, \quad d_5(p) = 0, \quad d_6(p) = 0, \qquad p \neq 3 \tag{5.17}$$
$$d_1(3) = 0, \quad d_2(3) = 0, \quad d_3(3) = 2, \quad d_4(3) = 0, \quad d_5(3) = 0, \quad d_6(3) = 0, \tag{5.18}$$

Therefore, the Jordan normal form of $A_{r,s}$ is given by

$$A_{r,s} \cong \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{if } p \neq 3, \quad A_{r,s} \cong \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{if } p = 3. \tag{5.19}$$

5.2 Computation of $A(t, \bar{t})$

Let $t = ((n_1, s_1), \ldots, (n_m, s_m))$ with $s_i = (s_{i,1}, \ldots, s_{i,m_i})$ be a type of a matrix. We use the notation from Section 5.1 and, thus, $g \in \mathrm{GL}(r, \mathbb{E})$ is a matrix in Jordan normal form similar to matrices in $\mathrm{GL}(r, q)$ of type t. Analogously, $\bar{g} \in \mathrm{GL}(r^2, \mathbb{E})$ is a matrix in Jordan normal form similar to $g \otimes g$. The type of $\bar{g}$ is $\bar{t}$.
Recall: We have $\mathcal{E}(t) = \{g \in \mathrm{GL}(r, q) \,|\, g \text{ has type } t\}$ and $\mathcal{O}(t) = \{g \otimes g \mid g \in \mathcal{E}(t)\}$. Let $\mathcal{R}(t)$ be a set of representatives for the conjugacy classes of matrices of type t. Then, $A(t, \bar{t})$ is the number of representatives $g \in \mathcal{R}(t)$ such that $g \otimes g \in \mathcal{O}(t)$ has type $\bar{t}$.
In this section we will give an algorithm that determines all types of matrices $\bar{t}$ in the set $\mathcal{O}(t)$ and computes the number $A(t, \bar{t})$. Hence, the following two questions must be answered:

1. Which types $\bar{t}$ occur in $\mathcal{O}(t)$?

2. How many conjugacy classes of type $\bar{t}$ exist in $\mathcal{O}(t)$?

In Section 5.1 we have seen that a matrix of type t with type-parameters (n_i, s_i) with $s_i = (s_{i,1}, \ldots, s_{i,m_i})$ for $1 \leq i \leq m$ can be represented by a matrix $g \in \mathrm{GL}(r, \mathbb{E})$ in Jordan normal form

$$g = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m_i} \bigoplus_{k=0}^{n_i - 1} J_{s_{i,j}} \left(a_i^{q^k} \right). \tag{5.20}$$

In Equation (5.20) the elements $a_1, \ldots, a_m \in \hat{\mathbb{E}}$ $(1 \leq i \leq m)$ and all their powers are pairwise distinct. Then, we apply the Kronecker product to g. Lemma 5.4 shows that the Kronecker product is compatible with the direct sum and we get that $g \otimes g$ is similar (over $\mathrm{GL}(r^2, \mathbb{E})$) to

$$\hat{g} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m_i} \bigoplus_{k=0}^{n_i - 1} \bigoplus_{i'=1}^{m} \bigoplus_{j'=1}^{m_{i'}} \bigoplus_{k'=0}^{n_{i'} - 1} J_{s_{i,j}} \left(a_i^{q^k} \right) \otimes J_{s_{i',j'}} \left(a_{i'}^{q^{k'}} \right). \tag{5.21}$$

Then, Theorem 5.5 shows how the Kronecker product of two Jordan blocks can be decomposed into Jordan blocks. It follows:

$$J_{s_{i,j}} \left(a_i^{q^k} \right) \otimes J_{s_{i',j'}} \left(a_{i'}^{q^{k'}} \right) \quad \sim_{\mathbb{E}} \quad \bigoplus_{\nu=1}^{s_{i,j} \cdot s_{i',j'}} \bigoplus_{z=1}^{d_i(p)} J_\nu \left(a_i^{q^k} a_{i'}^{q^{k'}} \right). \tag{5.22}$$

We use the similarity given in Equation (5.22) to replace the Kronecker products in Equation (5.21) and get a complete decomposition of $\hat{g}$ into a direct sum of Jordan blocks. Now, we need a way to read off the new type $\bar{t}$. The problem here is that even though all elements a_i and their powers are pairwise different it might happen that products $a_i^{q^k} a_{i'}^{q^{k'}}$ of two such elements will not be distinct anymore. Therefore, we define a set

$$D = D(t) = \left\{ a_i^{q^k} a_{i'}^{q^{k'}} \;\middle|\; 1 \leq i, i' \leq m,\, 0 \leq k < n_i,\, 0 \leq k' < n_{i'} \right\} \tag{5.23}$$

which coincides with the set of all eigenvalues of the Jordan blocks of $g \otimes g$. A possible equation amongst the eigenvalues can be identified with a pair (d_1, d_2) with elements $d_1, d_2 \in D$. Such pair (d_1, d_2) is formulated as an equation via $d_1 = d_2$. Hence, $D \times D$ can be interpreted as the set of all possible equations. There are some equations counted twice as the pairs (d_1, d_2) and (d_2, d_1) yield the same equation. Some equations are trivial, since pairs (d_1, d_1) yield the equation $d_1 = d_1$ that certainly holds.

Let $L = L(t)$ be the set of all possible equations between elements of $D = D(t)$. Then we have

$$L = L(t) = \left\{ a_i^{q^k} a_{i'}^{q^{k'}} = a_\iota^{q^\kappa} a_{\iota'}^{q^{\kappa'}} \;\middle|\; \text{all possible } i, i', k, k', \iota, \iota', \kappa, \kappa' \right\}. \tag{5.24}$$

At this point we recall the Frobenius automorphism and some of its properties (see for example [23]). Let $\mathbb{F}_q$ be a field of characteristic p. Then, the mapping $\varphi_p : \mathbb{F}_q \to \mathbb{F}_q$, $x \mapsto x^p$, is called Frobenius automorphism. One can check that the mapping is indeed an automorphism. Furthermore, the mapping $\varphi_q : \mathbb{F}_q \to \mathbb{F}_q$, $x \mapsto x^q$, is also an automorphism. Having a field extension $\mathbb{F}_{q^e}$ over $\mathbb{F}_q$, then φ_q fixes every element $x \in \mathbb{F}_q$. The Galois group of $\mathbb{F}_{q^e}/\mathbb{F}_q$ is cyclic and generated by φ_q.

By construction of $L(t)$ we get $a_i^{q^{n_i}} = a_i$ for $1 \leq i \leq m$. When writing the powers $a_i^{q^k}$ we always agree to choose the smallest possible $k \in \{0, \ldots, n_i - 1\}$ and for each i there is just a finite number of exponents possible.
It is possible that the eigenvalues satisfy more than one equation. Hence, we can take the power set of $L(t)$ to describe all combinations of possible equations amongst the eigenvalues in D.

Definition 5.7: *Galois closed subsets*
*A subset $U \subseteq L$ is called **Galois closed** if it is closed under taking q'th powers. Hence, $U \subseteq L$ is Galois closed if the following condition is satisfied for every equation in U:*

$$a_i^{q^k} a_{i'}^{q^{k'}} = a_\iota^{q^\kappa} a_{\iota'}^{q^{\kappa'}} \in U \quad \Longrightarrow \quad a_i^{q^{k+1}} a_{i'}^{q^{k'+1}} = a_\iota^{q^{\kappa+1}} a_{\iota'}^{q^{\kappa'+1}} \in U. \tag{5.25}$$

The set of all Galois closed subsets of L is denoted by $\mathcal{G}al(L)$.

It is simple to determine the set of Galois closed subsets, even though there might be a large number of those subsets. The next lemma points at a connection between the possible types $\bar{t}$ arising by the application of the Kronecker product and the Galois closed subsets.

Lemma 5.8 *Let t be a type of matrices in $\mathrm{GL}(r,q)$ and let $\mathcal{O}(t)$ and L be defined as above. For every type $\bar{t}$ of matrices in $\mathcal{O}(t)$ there exists a Galois closed subsets $U \in \mathcal{G}al(L)$ which completely determines the type t.*

Proof: Let $g \in \mathrm{GL}(r,q)$ be a matrix of type t and let (n_i, s_i) and $(n_{i'}, s_{i'})$ be two type-parameters of t. Using Theorem 5.5 we get natural numbers $d_1, \ldots, d_h$ all depending on $s_{i,j}$, $s_{i',j'}$, and on the characteristic of $\mathbb{E}$ such that

$$J_{s_{i,j}}(1) \otimes J_{s_{i',j'}}(1) = J_{d_1}(1) \oplus \ldots \oplus J_{d_h}(1). \tag{5.26}$$

By Lemma 5.4 $g \otimes g$ is conjugate over $\mathrm{GL}(r^2, \mathbb{E})$ to a block-diagonal matrix whose diagonal blocks are given by the summands

$$\bigoplus_{k=0}^{n_i-1} \bigoplus_{k'=0}^{n_{i'}-1} J_{s_{i,j}}(a_i^{q^k}) \otimes J_{s_{i',j'}}(a_{i'}^{q^{k'}}) \tag{5.27}$$

$$= \bigoplus_{k=0}^{n_i-1} \bigoplus_{k'=0}^{n_{i'}-1} a_i^{q^k} a_{i'}^{q^{k'}} J_{s_{i,j}}(1) \otimes J_{s_{i',j'}}(1) \tag{5.28}$$

$$= \bigoplus_{k=0}^{n_i-1} \bigoplus_{k'=0}^{n_{i'}-1} \bigoplus_{l=1}^{h} a_i^{q^k} a_{i'}^{q^{k'}} J_{d_l}(1) \tag{5.29}$$

$$=\bigoplus_{l=1}^{h}\left(\bigoplus_{k=0}^{n_i-1}\bigoplus_{k'=0}^{n_{i'}-1} a_i^{q^k} a_{i'}^{q^{k'}} J_{d_l}(1)\right) \tag{5.30}$$

$$=\bigoplus_{l=1}^{h}\left(\bigoplus_{k'=0}^{\gcd(n_i,n_{i'})-1}\left(\bigoplus_{k=0}^{\mathrm{lcm}(n_i,n_{i'})-1}(a_i a_{i'}^{q^{k'}})^{q^k} J_{d_l}(1)\right)\right). \tag{5.31}$$

Let us explain the last step (5.31) in the above equation. We abbreviate $a = a_i a_{i'}^{q^{k'}}$ and $l = \mathrm{lcm}(n_i, n_{i'})$. Then, $p(x) = (x-a)(x-a^q)\cdot\ldots\cdot(x-a^{q^{l-1}})$ is a polynomial in $\mathbb{F}_q[x]$. All its roots are given by $a_i^{q^k} a_{i'}^{q^{k'}}$ with $0 \le k \le n_i - 1$ and $0 \le k' \le n_i' - 1$. These are exactly the terms at step (5.30) in the above equation. Next, let $c \in \mathbb{N}$ be minimal such that $a^{q^c} = a$. Then, we can split $p(x)$ over $\mathbb{F}_q$ into l/c factors, each of the form

$$\left((x-a)\cdot\ldots\cdot(x-a^{q^{c-1}})\right)^{l/c}. \tag{5.32}$$

Now, Equation (5.32) states for the eigenvalue $a = a_i a_{i'}^{k'}$ a constant c with $a^{q^c} = a$. Doing this for all occurring products $a = a_i^k a_{i'}^{k'}$ we can describe the field extension containing those products. We then need to add all equations stating which of those products are equal. Let U be the set of all those equations.
The type $\bar{t}$ of $g \otimes g$ is then completely determined by the equations in the set U: The eigenvalues of $g \otimes g$ must satisfy all equations in U, but must not satisfy any equation in $L \setminus U$. By construction holds that U is a Galois closed subset of L. □

Given a type t of matrices in $\mathrm{GL}(r, q)$ we can determine D as in Equation (5.23) and deduce the set L of all possible equations amongst the elements in D. Having computed all possible Galois closed subsets $\mathcal{G}al(L)$ of L we can determine all possible types $\bar{t}$ in $\mathcal{O}(t)$. We write $\bar{t} = \bar{t}(U)$ if the type $\bar{t}$ of matrices in $\mathcal{O}(t)$ is determined by the Galois closed subset $U \in \mathcal{G}al(L)$. Here, we want to stress that the eigenvalues of a matrix of type $\bar{t} = \bar{t}(U)$ must satisfy all equations in U, but none of the equations in $L \setminus U$.
In a next step we investigate the number of representatives for the conjugacy classes of matrices of type t which yield the type $\bar{t}(U)$ for some $U \in \mathcal{G}al(L)$ by application of the Kronecker product. We do this for all $U \in \mathcal{G}al(L)$. For some Galois closed subsets $U \in \mathcal{G}al(L)$ it happens that there exist no elements satisfying the equations in U and none of the equations in $L \setminus U$. Such a subset U will not yield a type $\bar{t}(U)$ of matrices in $\mathcal{O}(t)$. The number of representatives is zero. Though, we use the symbol $A(t, \bar{t}(U))$ and we write $A(t, \bar{t}(U)) = 0$.
For a systematically investigation of $A(t, \bar{t}(U))$ we consider some further sets of equations. For this purpose for every degree n_i we define a set μ_i which contains all maximal proper divisors of n_i. That is

$$\mu_i = \{n_i/p \mid p \text{ is a prime dividing } n_i\}. \tag{5.33}$$

Furthermore, we define a set $I = I_1 \cup I_2$ with

$$I_1 = \left\{a_i = a_i^{q^l} \mid 1 \le i \le m \quad \text{and} \quad l \in \mu_i\right\}, \qquad \text{and} \tag{5.34}$$

$$I_2 = \left\{a_i = a_j^{q^k} \mid 1 \le i, j \le m \quad \text{with} \quad i \ne j \quad \text{and} \quad n_i = n_j \quad \text{for} \quad 0 \le k < n_j\right\}. \tag{5.35}$$

The first set I_1 contains equations stating that an element $a_i \in \mathbb{E}_i$ is already contained in some proper intermediate field between $\mathbb{F}_q$ and $\mathbb{E}_i$. The equations in I_2 state that two elements a_i and a_j which are roots of polynomials of the same degree $n_i = n_j$ are Galois conjugates of each other. This is equivalent to saying that the polynomials of degree $n_i = n_j$ are identical.
We see that none of the equations in I must be satisfied: An element a_i is always chosen to be in $\hat{\mathbb{E}}_i$ and, hence, it cannot come from any subfield of $\mathbb{E}_i$. Thus, the equations in I_1 must not be satisfied.

Furthermore, all polynomials are pairwise different. Hence, they cannot have common roots over $\mathbb{E}_i$ and their roots cannot be Galois conjugates.
For a Galois closed subset $U \in \mathcal{G}al(L)$ we define a set $K_I(U)$ as follows:

$$K_I(U) = \{a = (a_1, \ldots, a_m) \in \mathbb{E}_1 \times \ldots \times \mathbb{E}_m \mid a \text{ satisfies all equations in } U\text{, but none in } I\}\,. \quad (5.36)$$

The following lemma explains the connection between the desired number $A(t, \bar{t}(U))$ and the set $K_I(U)$.

Lemma 5.9 *Let $U \in \mathcal{G}al(L)$ and let $W_1, \ldots, W_r$ denote the minimal supersets of U in $\mathcal{G}al(L)$. Then,*

$$A(t, \bar{t}(U)) = \frac{1}{\pi(t) \cdot n_1 \cdots n_m} \left(|K_I(U)| + \sum_{l=1}^{r} (-1)^l \sum_{1 \le i_1 < \ldots < i_l \le r} |K_I(W_{i_1} \cup \ldots \cup W_{i_l})| \right). \quad (5.37)$$

The number $\pi(t)$ is defined as in the proof of Lemma 3.10 to be the number of permutations of the type-parameters of t that do not change the type t.

Proof: First of all we note the following fact: A maximal subfield $\mathbb{K}_i$ of $\mathbb{E}_i$ has degree $[\mathbb{E}_i : \mathbb{K}_i] = l \in \mu_i$ and the elements of $\mathbb{K}_i$ can be described by $\{a \in \mathbb{E}_i \mid a^{q^l} = a\}$. Each element of a proper subfield of $\mathbb{E}_i$ satisfies at least one of the equations in I_1.
Now, we consider an element $a = (a_1, \ldots, a_m) \in K_I(U)$. By construction a does not satisfy any equation in I_1 and it directly follows that $a = (a_1, \ldots, a_m) \in \hat{\mathbb{E}}_1 \times \ldots \times \hat{\mathbb{E}}_m$. Further, we have that a does not satisfy any equation in I_2 and therefore, we have that a_i and a_j (with $i \neq j$) can be considered as roots of distinct polynomials. We conclude that a defines a Jordan normal form of an element g of some type t as described in Theorem 5.2.
Furthermore, the element $a = (a_1, \ldots, a_m)$ satisfies all equations in the Galois closed subset U. Hence, the element $g \otimes g$ has the type $\bar{t}(W)$ for some $W \in \mathcal{G}al(L)$ with $W \supseteq U$. We use the inclusion-exclusion principle to ensure that only Jordan normal forms of elements g are counted such that $g \otimes g$ has type $\bar{t}(U)$.
As a last step we have to take care of permutations amongst the type-parameters. This gives us the factor $\frac{1}{\pi(t) \cdot n_1 \cdots n_m}$ as in Corollary 5.3. □

The previous Lemma 5.9 shows how to compute the number $A(t, \bar{t}(U))$ by determining the cardinality of the sets $K_I(U)$ for Galois closed subsets $U \in \mathcal{G}al(L)$. It remains to determine the number of elements in $K_I(U)$. We use an inclusion-exclusion principle again: For every subset $J \subseteq I$ we define a set $\breve{K}_J(U)$ by

$$\breve{K}_J(U) = \{a = (a_1, \ldots, a_m) \in \mathbb{E}_1 \times \ldots \times \mathbb{E}_m \mid a \text{ satisfies all equations in } U \text{ and in } J\}\,. \quad (5.38)$$

Lemma 5.10 *Let $U \in \mathcal{G}al(L)$. Then,*

$$|K_I(U)| = \sum_{J \subseteq I} (-1)^{|J|} |\breve{K}_J(U)|. \quad (5.39)$$

Proof: This is a direct application of the inclusion-exclusion principle. □

We have reduced the determination of $A(t, \bar{t}(U))$ to the determination of $\breve{K}_J(W)$ for all $J \subseteq I$ and all $W \supseteq U$. This final step is provided by the following theorem. We use its constructive proof and translate it into an algorithm.

Theorem 5.11 ***(Higman [18], Vaughan-Lee [34])***
Let $U \subseteq L$ and $J \subseteq I$. Then, $|\breve{K}_J(U)|$ can be described by a PORC polynomial.

Proof: We follow Vaughan-Lee who gives in [34] a constructive proof for this theorem which is based on Higman (see [18, Sec. 2.2]). For completeness, its main steps are recalled briefly.
All elements in $\breve{K}_J(U)$ are described by the equations in J, in U, and additionally in H, which is given by

$$H = \left\{ a_i^{q^{n_i}} = a_i \quad \text{for} \quad 1 \le i \le m \right\}. \tag{5.40}$$

The equations in H ensure that the elements in a_i are contained in $\mathbb{E}_i$. Solving all equations for 1 on the right hand side yields:

$$a_i^{q^k} a_{i'}^{q^{k'}} a_\iota^{-q^\kappa} a_{\iota'}^{-q^{\kappa'}} = 1 \qquad \text{(equations in } U\text{)}, \tag{5.41}$$

$$a_i a_j^{-q^k} = 1 \qquad \text{(equations in } J\text{)}, \tag{5.42}$$

$$a_i^{q^{n_i}-1} = 1 \qquad \text{(equations in } H\text{)}. \tag{5.43}$$

Hence, all equations in $U \cup J \cup H$ can be written in a general form $a_1^{e_1(q)} \cdot \ldots \cdot a_m^{e_m(q)}$ with polynomials $e_1(q), \ldots, e_m(q) \in \mathbb{Z}[q]$. In the terms of Higman such equations are called *monomial equations*.
We define $\breve{M}_J(U)$ as the matrix whose rows consists of the vectors $(e_1(q), \ldots, e_m(q)) \in \mathbb{Z}[q]^m$ corresponding to $U \cup J \cup H$. Thus, all information to completely determine $|\breve{K}_J(U)|$ is given by $\breve{M}_J(U)$. Using the equations in H ensures that $\breve{M}_J(U)$ has rank m.
Now, let $m_1(q), \ldots, m_z(q)$ be the matrix minors of order m. Hence, every $m_i(q)$ is the determinant of some $m \times m$-submatrix of $\breve{M}_J(U)$ and it is $m_1(q), \ldots, m_z(q) \in \mathbb{Z}[q]$. We define the function $f(q)$ to be the point-wise greatest common divisor of the $m_i(q)$, hence, for each prime power $c \in \mathbb{N}$ it is $f(c) = \gcd(m_1(c), \ldots, m_z(c))$.
Higman's main result at this point is (see [18, Theorem 2.2.2]) that the number of elements which have to satisfy a certain set of (monomial) equations is PORC. Vaughan-Lee [34] proves that this number agrees with the point-wise greatest common divisor of the matrix minors of order m. His main argument is: When bringing the matrix $\breve{M}_J(I)$ into Smith normal form (over $\mathbb{Z}[q]$) the terms within the matrix will change, but not the number of elements described by those terms. As the Smith normal form can be computed using the matrix minors the number of solutions can be given as the point-wise greatest common divisor of the minors. In Section 2.2 we have shown that the point-wise greatest common divisor of integer polynomials is PORC, so we agree with the results of Higman and Vaughan-Lee.
In [34], Vaughan-Lee gives an algorithm to compute the point-wise greatest common divisor. We have already discussed this algorithm in Section 2.2. It is summarised as Algorithm 3 on page 104. □

Theorem 5.11 completes the algorithm to determine the number $A(t, \bar{t}(U))$ for all types $\bar{t}(U)$ defined by Galois closed subsets $U \in \mathcal{G}al(L)$ and, hence, (with Lemma 5.8) for all types $\bar{t}$ of matrices in $\mathcal{O}(t)$. We point at the following corollary which now directly follows from Theorem 5.11.

Corollary 5.12 *The numbers $A(t, \bar{t})$, considered as a function in q, are PORC.*

One can find the algorithm for the computation of the numbers $A(t, \bar{t})$ summarised in Algorithm 7 (see Appendix B). We will close this section with an example of this algorithm.

Example 12. We take the type $t = ((2,(1)))$ of an element in $\mathrm{GL}(2,q)$. A matrix $\tilde{g}$ of type t can be represented over $\mathbb{F}_q$ by the companion matrix of an irreducible polynomial of degree 2. The splitting field $\mathbb{E}$ is a field extension of degree 2 over $\mathbb{F}_q$ and, thus, the Jordan normal form of $\tilde{g}$ over $\mathbb{E}$ is g, where g is a diagonal matrix with diagonal entries a_1 and a_1^q for some $a_1 \in \hat{\mathbb{E}} = \mathbb{E} \setminus \mathbb{F}_q$. By construction we also have $a_1^q \in \hat{\mathbb{E}}$. To shorten notation we will omit the index 1 all the times as here is just one polynomial and, hence, one root $a = a_1$ to be chosen.
Thus, g is a matrix over $\mathbb{E}$ which is in Jordan normal form and represents the type t. By $\bar{g}$ we denote the matrix in Jordan normal form over $\mathbb{E}$ conjugate to $g \otimes g$ over $\mathrm{GL}(4,\mathbb{E})$. Then g and $\bar{g}$ are

$$g = \begin{pmatrix} a & 0 \\ 0 & a^q \end{pmatrix} \qquad \text{and} \qquad \bar{g} = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & a^{q+1} & 0 & 0 \\ 0 & 0 & a^{q+1} & 0 \\ 0 & 0 & 0 & a^{2q} \end{pmatrix} \tag{5.44}$$

Since we choose $a \in \hat{\mathbb{E}}$ we have $q^2 - q$ choices for a. When choosing a, the element a^q is also completely determined. But the ordering of a and a^q within our matrices does not matter and, therefore, the number of different conjugacy classes for matrices of type t is $k_t(q) = (q^2 - q)/2$.
Now, we consider the set D of eigenvalues of the Jordan normal form $\bar{g}$. We get

$$D = \{a^2, a^{q+1}, a^{2q}\}. \tag{5.45}$$

The set $L = L(t)$ of equations can be given by:

$$L = \left\{a^2 = a^{q+1},\, a^2 = a^{2q},\, a^{q+1} = a^{2q}\right\}. \tag{5.46}$$

To determine $\mathcal{G}al(L)$ we take the power set of L and then test which elements are Galois closed subsets of L. The Galois closed subsets are

$$\mathcal{G}al(L) = \left\{\emptyset,\, \left\{a^2 = a^{2q}\right\},\, \left\{a^2 = a^{q+1}, a^{q+1} = a^{2q}\right\},\, L\right\}. \tag{5.47}$$

To shorten notation we will write $\mathcal{G}al(L) = \{U_1, U_2, U_3, U_4\}$ where we use the same ordering for the sets U_i as in Equation (5.47).
It remains to determine the sets $I = I_1 \cup I_2$ and H. We need to choose $a \in \mathbb{E} = \mathbb{F}_{q^2}$ such that $a \notin \mathbb{F}_q$. Hence, we obtain $I_1 = \{a^q = a\}$ and $H = \{a^{q^2} = a\}$. As there is just a single type-parameter, we cannot extend I_1 and H by any more equations and no incidences between roots of different polynomials can occur. This gives $I_2 = \emptyset$. Summarised we have

$$H = \{a^{2q} = a\} \qquad \text{and} \qquad I = \{a^q = a\}. \tag{5.48}$$

Now, we apply Lemma 5.9 to this situation. For every Galois closed subset $U \in \mathcal{G}al(L)$ we determine all minimal supersets within $\mathcal{G}al(L)$ and find $|K_I(U)|$ for all $U \in \mathcal{G}al(L)$. By arranging all elements in $\mathcal{G}al(L)$ as a graph, see Figure 5.1, we can read off whether or not a Galois closed set is contained in another one.
It remains to determine $|\breve{K}_J(U)|$ for all $U \in \mathcal{G}al(L)$ and for all $J \subseteq I$. This will be done by computing the matrix minors of $\breve{M}_J(U)$ for all possible combinations of U and J.
We just have two subsets of I, namely $J = \emptyset$ and $J = I$. Therefore, let $\breve{M}_I(U)$ denote the matrix with all equations in $U \cup H \cup I$ and let $\breve{M}_\emptyset(U)$ denote the matrix with all equations in $U \cup H$.

$$
\begin{tikzcd}
 & L = U_4 \arrow[dl] \arrow[dr] & \\
\{a^2 = a^{2q}\} = U_2 \arrow[dr] & & U_3 = \{a^2 = a^{q+1}, a^{q+1} = a^{2q}\} \arrow[dl] \\
 & U_1 = \emptyset &
\end{tikzcd}
$$

Figure 5.1: The sets of $\mathcal{G}al(L)$ arranged in a graph. An arrow $V \to W$ denotes that V is a minimal superset of W.

- We start with $U_4 = L$. As we have just one column, we can read off the minors: They are the elements within the matrices.

$$M_\emptyset(U_4) = \begin{pmatrix} q^2-1 \\ q-1 \\ 2q-2 \\ -q+1 \end{pmatrix} \quad \text{and} \quad M_I(U_4) = \begin{pmatrix} q^2-1 \\ q-1 \\ q-1 \\ 2q-2 \\ -q+1 \end{pmatrix}. \tag{5.49}$$

In both cases the point-wise greatest common divisor over all matrix entries is the function $q-1$ and, hence, we have $|\breve{K}_\emptyset(U_4)| = q-1 = |\breve{K}_I(U_4)|$. Thus, the inclusion-exclusion principle given by Lemma 5.10 yields

$$|K_I(U_4)| = |\breve{K}_\emptyset(U_4)| - |\breve{K}_I(U_4)| = 0. \tag{5.50}$$

- For the set U_3 we have the matrices

$$M_\emptyset(U_3) = \begin{pmatrix} q^2-1 \\ q-1 \\ -q+1 \end{pmatrix} \quad \text{and} \quad M_I(U_3) = \begin{pmatrix} q^2-1 \\ q-1 \\ q-1 \\ -q+1 \end{pmatrix} \tag{5.51}$$

and analogously to the previous case we have $|\breve{K}_\emptyset(U_3)| = q-1 = |\breve{K}_I(U_3)|$ which yields

$$|K_I(U_3)| = |\breve{K}_\emptyset(U_3)| - |\breve{K}_I(U_3)| = 0. \tag{5.52}$$

- For the set U_2 we get the matrices

$$M_\emptyset(U_2) = \begin{pmatrix} q^2-1 \\ 2q-2 \end{pmatrix} \quad \text{and} \quad M_I(U_2) = \begin{pmatrix} q^2-1 \\ q-1 \\ 2q-2 \end{pmatrix}. \tag{5.53}$$

As above we get $|\breve{K}_I(U_2)| = q - 1$, but the case $J = \emptyset$ is more interesting. As in Example 3 on page 9 we have $|\breve{K}_\emptyset(U_2)| = (q-1) \cdot \gcd(q-1, 2)$. This gives $|K_I(U_2)| = q - 1$ if q is odd and $|K_I(U_2)| = 0$ otherwise. Combined to a PORC function we have

$$|K_I(U_2)| = |\breve{K}_\emptyset(U_2)| - |\breve{K}_I(U_2)| = q \cdot (\gcd(q-1, 2) - 1) - \gcd(q-1, 2) + 1. \tag{5.54}$$

- We are left with the case U_1 which gives the matrices

$$M_\emptyset(U_1) = \begin{pmatrix} q^2 - 1 \end{pmatrix} \qquad \text{and} \qquad M_I(U_1) = \begin{pmatrix} q^2 - 1 \\ q - 1 \end{pmatrix}. \tag{5.55}$$

Hence, we have $|\breve{K}_\emptyset(U_1)| = q^2 - 1$ and $|\breve{K}_I(U_2)| = q - 1$. Thus, we obtain

$$|K_I(U_1)| = |\breve{K}_\emptyset(U_1)| - |\breve{K}_I(U_1)| = q^2 - 1 - (q - 1) = q^2 - q. \tag{5.56}$$

We use Lemma 5.9 to compute the numbers $A(t, \bar{t}(U))$ for the four Galois closed subsets $U_1, \dots, U_4$. The type t directly gives $\pi(t) = 1$ as there is just one type-parameter $(2, (1))$. We also read off $n_1 = 2$. All in all, the coefficients outside the product terms in Equation (5.37) evaluate to $\frac{1}{2}$. It follows

$$A(t, \bar{t}(U_4)) = \tfrac{1}{2}\left(K_I(U_4)\right) = 0, \tag{5.57}$$
$$A(t, \bar{t}(U_3)) = \tfrac{1}{2}\left(K_I(U_3) - K_I(U_4)\right) = 0, \tag{5.58}$$
$$A(t, \bar{t}(U_2)) = \tfrac{1}{2}\left(K_I(U_2) - K_I(U_4)\right) = q \cdot \left(\tfrac{\gcd(q-1,2)-1}{2}\right) - \left(\tfrac{\gcd(q-1,2)-1}{2}\right), \tag{5.59}$$
$$\begin{aligned} A(t, \bar{t}(U_1)) &= \tfrac{1}{2}\left(K_I(U_1) - K_I(U_2) - K_I(U_3) + K_I(U_4)\right) \\ &= \tfrac{1}{2} \cdot \left(q^2 - q - q \cdot (\gcd(q-1, 2) - 1) + \gcd(q-1, 2) - 1\right) \\ &= q^2 \cdot \tfrac{1}{2} + \left(\tfrac{\gcd(q-1,2)-2}{2}\right) \cdot q - \left(\tfrac{\gcd(q-1,2)}{2}\right). \end{aligned} \tag{5.60}$$

It is reasonable that we have $A(t, \bar{t}(U_3)) = 0 = A(t, \bar{t}(U_4))$. The sets U_3 and U_4 both contain the equations $a^2 = a^{q+1}$ and $a^{q+1} = a^{2q}$. Both equations can be simplified to $a = a^q$ which contradicts the fact that a is chosen from $\hat{\mathbb{E}} = \mathbb{F}_{q^2} \setminus \mathbb{F}_q$.

5.3 Notes on the algorithm for $A(t, \bar{t})$

The most time-consuming part of our algorithm is the determination of all types $\bar{t}$ in $\mathcal{O}(t)$ and the corresponding numbers $A(t, \bar{t})$ of conjugacy classes of type $\bar{t}$.
For instance, for a fixed rank r we always consider the type t with r type-parameters $(1, (1))$. This is the case of a matrix g with r distinct eigenvalues $a_1, \dots, a_r$. The eigenvalues of $g \otimes g$ are given by

$D(t) = \{a_i a_j \mid 1 \leq i \leq j \leq r\}$ with $|D(t)| = \frac{r(r+1)}{2}$. The set of equations that can be formulated by elements of $D(t)$ is given by $L(t)$ and it follows

$$|L(t)| = \binom{|D(t)|}{2} = \tfrac{1}{8}(r-1)r(r+1)(r+2). \tag{5.61}$$

Since all elements in $D(t)$ come from the ground field $\mathbb{F}_q$, in this case the set $\mathcal{G}al(L)$ of Galois closed subsets of $L(t)$ coincides with the powerset of $L(t)$. Given an element $U \in \mathcal{G}al(L)$ let $\bar{t} = t(U)$ be the type of matrices in $\mathcal{O}(t)$ determined by U. The number $A(t, \bar{t})$ is the number of solutions when choosing the elements in $D(t)$ restricted by all equations in U. We can estimate the runtime of the algorithm, measured in the number of sets of equations considered, by $\mathcal{O}(2^{r^4})$ since this case yields most equations. Hence, the runtime increases significantly in r.

Since we have a huge amount of equations that must be considered, an important issue of our implementation is the reduction of the runtime. Therefore, we use as many symmetries among the eigenvalues as possible: It might happen that two or more type-parameters of some type are equal, say, it is $(n_i, s_i) = (n_j, s_j)$. Let a_i be a root of an irreducible polynomial of degree n_i arising from (n_i, s_i) and let a_j be a root of a different irreducible polynomial of degree $n_i = n_j$ which is defined by (n_j, s_j). In all our computation we need to consider a_i and a_j to be different and that they are no conjugates of each other. Apart from that a_i and a_j can freely be swapped in all our computations. The symmetry among the eigenvalues leads to a symmetry among the set of equations L and $\mathcal{G}al(L)$.
We first determine the symmetric group respecting the symmetries S_E among the eigenvalues. Then, we compute the induced symmetric group S_L describing the symmetries among the equations in L and $\mathcal{G}al(L)$, respectively. We determine the orbits of S_L acting on L and we take one representative for each orbit that is contained in $\mathcal{G}al(L)$. Instead of continuing our computation with all equations in $\mathcal{G}al(L)$ we go on only using the chosen representatives.
Another technical improvement is to use information of subsets of L that do not lead to any solutions. Say that there remain six representative equations in L after we have used the group action. If for instance the equations 2 and 3 yield together no solution then any subset of L containing equations 2 and 3 will not yield any solution either.
Let us consider the following example to see how these improvements of our algorithm work.

Example 13. Let us take the type $t = ((1,(1)),(1,(1)),(1,(1)))$ of matrices in $\mathrm{GL}(3,q)$. The Jordan normal form g of a matrix of type t is a diagonal matrix with three distinct eigenvalues $a_1, a_2, a_3 \in \mathbb{F}_q$. For our computation we can consider those three eigenvalues to be similar. The symmetric group on the eigenvalues is $S_E = S_3$ acting on the set of eigenvalues via permutation.
Before we are able to benefit from this permutation group we determine the sets D and L. For D it follows:

$$D = D(t) = \{a_1^2, a_1a_2, a_1a_3, a_2^2, a_2a_3, a_3^2\}. \tag{5.62}$$

For the determination of all equations amongst the elements of D we refer to Table 5.2. There, all possible pairs of elements of D are formulated as equations and all impossible equations are cancelled. It remains a set L of six equations.

$$L = \{a_1^2 = a_2^2, a_1^2 = a_2a_3, a_1^2 = a_3^2, a_3^2 = a_1a_2, a_2^2 = a_1a_3, a_2^2 = a_3^2\}. \tag{5.63}$$

Let us investigate which of the equations behave the same: We have the symmetric group S_3 acting on the set of eigenvalues. We translate this action to an action on L and, hence, we look for a suitable subgroup of S_6.

	a_1^2	a_1a_2	a_1a_3	a_2^2	a_2a_3	a_3^2
a_1^2	–	$\cancel{a_1^2 = a_1a_2}$	$\cancel{a_1^2 = a_1a_3}$	$a_1^2 = a_2^2$	$a_1^2 = a_2a_3$	$a_1^2 = a_3^2$
a_1a_2	$\cancel{a_1^2 = a_1a_2}$	–	$\cancel{a_1a_2 = a_1a_3}$	$\cancel{a_1a_2 = a_2^2}$	$\cancel{a_1a_2 = a_2a_3}$	$a_1a_2 = a_3^2$
a_1a_3	$\cancel{a_1^2 = a_1a_3}$	$\cancel{a_1a_2 = a_1a_3}$	–	$a_1a_3 = a_2^2$	$\cancel{a_1a_3 = a_2a_3}$	$\cancel{a_1a_3 = a_3^2}$
a_2^2	$a_1^2 = a_2^2$	$\cancel{a_1a_2 = a_2^2}$	$a_1a_3 = a_2^2$	–	$\cancel{a_2^2 = a_2a_3}$	$a_2^2 = a_3^2$
a_2a_3	$a_1^2 = a_2a_3$	$\cancel{a_1a_2 = a_2a_3}$	$\cancel{a_1a_3 = a_2a_3}$	$\cancel{a_2^2 = a_2a_3}$	–	$\cancel{a_2a_3 = a_3^2}$
a_3^2	$a_1^2 = a_3^2$	$a_1a_2 = a_3^2$	$\cancel{a_1a_3 = a_3^2}$	$a_2^2 = a_3^2$	$\cancel{a_2a_3 = a_3^2}$	–

Table 5.2: A table showing how to obtain the set of all equations. First of all, any pair of two distinct eigenvalues yields an equation. Along the main diagonal we do not obtain an equation. As "=" is symmetric it is sufficient just to consider the upper or lower triangular block, for instance the upper one. Then, all inconsistent equations are cancelled. Just six equations remain.

Index number	Equation	Index number	Equation
1	$a_1^2 = a_2^2$	2	$a_1^2 = a_2a_3$
6	$a_2^2 = a_3^2$	5	$a_2^2 = a_1a_3$
3	$a_1^2 = a_3^2$	4	$a_3^2 = a_1a_2$

Table 5.3: The possible equations of L together with their index number.

We fix an order of the equations as it is given in Table 5.2 (reading each row from left to right) which is the same order as in Equation (5.63). The order makes it is easier to write down the permutations of the equations. The equations and their indexing numbers are summarised in Table 5.3.
For example, let us take the permutation $\sigma \in S_3$ which changes a_1 and a_2 and fixes a_3. The equations $a_1^2 = a_2^2$ and $a_1a_2 = a_3^2$ are invariant under σ, but applying σ to $a_1^2 = a_2a_3$ yields $a_2^2 = a_1a_3$ and applying σ to $a_1^2 = a_3^2$ yields $a_2^2 = a_3^2$. It follows that σ translates to the permutation $(2,5)(3,6) \in S_6$ as it changes the equations 2 and 5 and the equations 3 and 6.
In the same way we can investigate the permutation τ where τ is defined as the cyclic permutation $a_1 \mapsto a_2 \mapsto a_3 \mapsto a_1$. The resulting permutation is $(1,6,3)(2,5,4)$. We know that σ and τ generate the group $S_E = S_3$. Thus, the group S_L is generated by $(1,6,3)(2,5,4)$ and $(2,5)(3,6)$ and it is a homomorphic image of S_3 in S_6, where the group S_6 can be seen as the group of all possible permutations of equations in L.
Now, the numbers $K_I(U)$ have to be computed. We directly see that every subgroup of L is Galois closed. We begin with Galois closed subsets of L of size 0 and, then, start an induction on the size. The set I contains the restrictions that the elements a_1, a_2, and a_3 are distinct. For more details on the computation of the numbers $K_I(U)$ we refer to Example 12. Here, we emphasise how to use the symmetry amongst the subsets of L to use as little subsets as possible.

- There is just one subset of size 0 and it is the empty set. For $U = \emptyset$ we obtain

$$K_I(\emptyset) = (q-1)(q-2)(q-3) = q^3 - 6q^2 + 11q - 6. \tag{5.64}$$

- We consider the subsets of size one. There are six such sets possible. We use the permutation group S_L and under its action there are two orbits: The equations 1, 6, and 3 form one orbit. The other orbit contains the equations 2, 4, and 5. In Table 5.3 the equations are already arranged in these orbits. Hence, it is sufficient to consider only the subsets $U_{\{1\}}$ and $U_{\{2\}}$

containing the equation with number 1 and 2, respectively. We obtain

$$K_I(U_{\{1\}}) = q^2 \cdot [2 - (q,2)] + 4q \cdot [(q,2) - 2] - [3(q,2) - 6], \tag{5.65}$$
$$K_I(U_{\{2\}}) = q^2 + q \cdot [(q,2) - 5] - [(q,2) - 4]. \tag{5.66}$$

- Now, we consider the subsets of L having size two. There are $\binom{6}{2} = 15$ such sets possible. We let the permutation group act on the set of pairs of equations. We obtain four orbits. The number i denotes equation number in L.

$$\begin{aligned} &\{\{1,2\},\{5,6\},\{3,4\},\{1,5\},\{2,3\},\{4,6\}\}, && \{\{1,3\},\{1,6\},\{3,6\}\}, \\ &\{\{1,4\},\{2,6\},\{3,5\}\}, && \{\{2,4\},\{2,5\},\{4,5\}\}. \end{aligned} \tag{5.67}$$

As representatives we take the sets $\{1,2\}$, $\{1,3\}$, $\{1,4\}$ and $\{2,4\}$. The corresponding sets of equations yield the following numbers:

$$K_I(U_{\{1,2\}}) = 0, \tag{5.68}$$
$$K_I(U_{\{1,3\}}) = 0, \tag{5.69}$$
$$K_I(U_{\{1,4\}}) = q \cdot [(q,2) + (q-1,4) - 3] - [(q,2) + (q-1,4) - 3], \tag{5.70}$$
$$K_I(U_{\{2,4\}}) = q \cdot [(q-1,3) - 1] - [(q-1,3) - 1]. \tag{5.71}$$

The (indexing) sets $\{1,2\}$ and $\{1,3\}$ yield no possible choices. This is indeed true: Equation 1 and 2 give $a_1^2 = a_2^2 = a_2 a_3$ which is equivalent to saying $a_2 = a_3$. Equation 1 and 3 give $a_1^2 = a_2^2 = a_3^2$ which is equivalent to saying that at least two of the values a_1, a_2 and a_3 are equal. So, from now, we can avoid all subsets of L which include $\{1,2\}$ or $\{1,3\}$. Furthermore, all subsets of L containing any element of the orbits of $\{1,2\}$ and $\{1,3\}$ can be excluded. Thus, already nine pairs of equations yield no solution.

- We now consider the sets of three elements. The above computation showed: We cannot take an equation from the left column in Table 5.3 together with an equation from the right column. Furthermore, we cannot take two equations from the left column. When taking at least three equations all of them must be chosen from the right column. It remains one possible indexing set: $\{2,4,5\}$. It follows:

$$K_I(U_{\{2,4,5\}}) = q \cdot [(q-1,3) - 1] - [(q-1,3) - 1]. \tag{5.72}$$

- It is not possible to choose any subsets of L with four elements: At least one of the equations must be from the left column in Table 5.3 and one equation must be from the right. We have already seen that this will not produce any solution.

It remains to apply the inclusion-exclusion principle. This is done as in Lemma 5.9 and so we use its notation. In our situation we have $\pi(t) = 3! = 6$ as all three eigenvalues can be swapped freely. Further, we have $n_1 = n_2 = n_3 = 1$.
For example, when determining $A(t, \bar{t}(\emptyset))$, we need to consider all subsets because the empty set is a subset of every set. We must not forget that even if we have just considered one representative for each orbit, we still need to count all subsets of the orbit. Therefore, we get the prefactors of the

$K_I(U)$ terms. We omit to write down all numbers $K_I(U)$ that are equal to 0. It follows

$$\begin{aligned} A(t,\bar{t}(\emptyset)) = \tfrac{1}{6}\Big(&K_I(\emptyset) - 3K_I(U_{\{1\}}) - 3K_I(U_{\{2\}}) \\ &+3K_I(U_{\{1,4\}}) + 3K_I(U_{\{2,4\}}) - K_I(U_{\{2,4,5\}})\Big) \end{aligned} \tag{5.73}$$

$$\begin{aligned} = q^3/6 + q^2 \cdot &[3(q,2) - 15]/6 \\ &+ q \cdot [-12(q,2) + 2(q-1,3) + 3(q-1,4) + 39]/6 \\ &+ [9(q,2) - 2(q-1,3) - 3(q-1,4) - 25]/6. \end{aligned} \tag{5.74}$$

Doing this for the remaining seven representatives of our orbits we obtain

$$\begin{aligned} A(t,\bar{t}(U_{\{1\}})) = q^2 \cdot [-(q,2)+2]/6 + q \cdot [3(q,2) - (q-1,4) - 5]/6 \\ + [-2(q,2) + (q-1,4) + 3]/6, \end{aligned} \tag{5.75}$$

$$\begin{aligned} A(t,\bar{t}(U_{\{2\}})) = q^2/6 + q \cdot [-(q-1,3) - (q-1,4) - q]/6 \\ + [(q-1,3) + (q-1,4)]/6, \end{aligned} \tag{5.76}$$

$$A(t,\bar{t}(U_{\{1,2\}})) = 0, \tag{5.77}$$

$$A(t,\bar{t}(U_{\{1,3\}})) = 0, \tag{5.78}$$

$$A(t,\bar{t}(U_{\{1,4\}})) = q \cdot [(q,2) + (q-1,4) - 3]/6 + [-(q,2) - (q-1,4) + 3]/6, \tag{5.79}$$

$$A(t,\bar{t}(U_{\{2,4\}})) = 0, \tag{5.80}$$

$$A(t,\bar{t}(U_{\{2,4,5\}})) = q \cdot [(q-1,3) - 1]/6 + [-(q-1,3) + 1]/6. \tag{5.81}$$

We see that the type $t = ((1,(1)),\ (1,(1)),\ (1,(1)))$ yields five types of matrices in $\mathcal{O}(t)$.

Chapter 6

The case $\mathrm{GL}(2,q)$

We use this section to close this thesis and to give an explicit proof of Theorem 1.5. We follow the algorithm exhibited in this thesis and, therefore, this proof illustrates an application of our algorithm.
We start with the determination of the set T of all possible types of matrices in $\mathrm{GL}(2,q)$ and for every type $t \in T$ we compute the associated centraliser order $c_t = c_t(q)$ and the number $k_t = k_t(q)$ of different conjugacy classes of elements in $\mathcal{E}(t)$. We recall that $\mathcal{E}(t) = \{g \in \mathrm{GL}(2,q) \mid g \text{ has type } t\}$ is the set of all matrices g of type t.
The following Lemma 6.1 summarises all this information.

Lemma 6.1 *Let q be an arbitrary prime power.*

1. *The type t $= ((2,(1)))$ yields $c_t(q) = q^2 - 1$ and $k_t(q) = (q^2 - q)/2$.*
2. *The type t $= ((1,(2)))$ yields $c_t(q) = q^2 - q$ and $k_t(q) = q - 1$.*
3. *The type t $= ((1,(1,1)))$ yields $c_t(q) = (q^2 - 1)(q^2 - q)$ and $k_t(q) = q - 1$.*
4. *The type t $= ((1,(1)),(1,(1)))$ yields $c_t(q) = (q-1)^2$ and $k_t(q) = (q-1)(q-2)/2$.*

Proof:

1. Let g be an element in $\mathrm{GL}(2,q)$ of type $((2,(1)))$. Then g has an irreducible minimal polynomial of degree 2, say $\mu_g(x) = x^2 + mx + n$. Note, that $n \neq 0$ must hold. Now, we want to compute the order of the centraliser of the element g in $\mathrm{GL}(2,q)$. Let h be any 2×2 matrix defined over $\mathbb{F}_q$ and let $a,b,c,d \in \mathbb{F}_q$. Then, we can write

$$g = \begin{pmatrix} 0 & -n \\ 1 & -m \end{pmatrix} \qquad \text{and} \qquad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.1}$$

   If h is an element of the centraliser of g in $\mathrm{GL}(2,q)$, then $gh = hg$ holds. This yields that the matrix h only depends on two values:

$$h = h_{a,b} = \begin{pmatrix} a & b \\ -\frac{b}{n} & \frac{an+bm}{n} \end{pmatrix} \qquad \text{with} \qquad \det(h_{a,b}) = \tfrac{1}{n} \cdot \left(b^2 + abm + a^2 n\right). \tag{6.2}$$

If we choose $a = 0$ it follows $\det(h_{0,b}) = b^2/n$. As we want $h_{a,b} \in \mathrm{GL}(2,q)$ we have to choose $b \neq 0$. If we choose $a \neq 0$, then we obtain

$$\det(h_{a,b}) = \tfrac{a^2}{n} \cdot \left(\left(\tfrac{b}{a}\right)^2 + \tfrac{b}{a} \cdot m + n \right). \tag{6.3}$$

The first factor is different from 0. The second factor is the value of the irreducible polynomial x^2+mx+n at the point $x = b/a$ and, hence, it is neither zero. Summarised, $h_{a,b}$ has a multiplicative inverse in $\mathrm{GL}(2,q)$ as long as we do not choose both a and b to be 0. Thus, we have $q^2 - 1$ choices for the values a and b and the centraliser order of g is $c_t = c_t(q) = q^2 - 1$.

As a last step, we want to compute the number $k_t(q)$. Let $\mathbb{E}$ be the field with q^2 elements. Then, there exists an element $a \in \mathbb{E} \setminus \mathbb{F}_q$ such that g is conjugate in $\mathrm{GL}(2,\mathbb{E})$ to

$$\begin{pmatrix} a & 0 \\ 0 & a^q \end{pmatrix}. \tag{6.4}$$

We follow the proof of Corollary 5.3 which gives the uniqueness of the set $\{a, a^q\}$ for the conjugacy class of the element g. The elements a and a^q must be chosen from $\mathbb{E}\setminus\mathbb{F}_q$. The order of the elements in the set $\{a, a^q\}$ does not matter, so we get $k_t(q) = (q^2 - q)/2$.

2. Now, let g be a matrix in $\mathrm{GL}(2,q)$ of type $((1,(2)))$. Thus, g has a minimal polynomial of the form $(x-a)^2$ for some $a \in \mathbb{F}_q^*$. We know that g is conjugate over $\mathrm{GL}(2,q)$ to the matrix

$$\begin{pmatrix} a & a \\ 0 & a \end{pmatrix}. \tag{6.5}$$

So, the element a completely determines the conjugacy class of g and we get $k_t(q) = q - 1$.
The centraliser of an element g consists of the matrices of the form

$$\begin{pmatrix} u & w \\ 0 & u \end{pmatrix}. \tag{6.6}$$

We obtain this form by using the same arguments as in the first part of the proof and see that u must not be 0, but that there are no more restrictions to the choice of u and w. Hence, we have $u \in \mathbb{F}_q^*$ and $w \in \mathbb{F}_q$ and thus, $c_t(q) = (q-1)q$.

3. Let g be an element of type $((1,(1,1)))$ in $\mathrm{GL}(2,q)$. Its minimal polynomial has the form $(x-a)$ for some $a \in \mathbb{F}_q^*$. Hence, g is conjugate in $\mathrm{GL}(2,q)$ to the diagonal matrix

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}. \tag{6.7}$$

It is well known that an element of this form lies in the centre of the group $\mathrm{GL}(2,q)$ and we conclude $c_t(q) = |\mathrm{GL}(2,q)| = (q^2-1)(q^2-q)$.

Furthermore, all conjugacy classes are completely determined by the element $a \in \mathbb{F}_q^*$ and we get $k_t(q) = |\mathbb{F}_q^*| = q - 1$.

4. At last, let $g \in \mathrm{GL}(2,q)$ be a matrix of type $((1,(1)),(1,(1)))$ which has a minimal polynomial of the form $(x-a)(x-b)$ for $a,b \in \mathbb{F}_q^*$ with $a \neq b$. We know that g is conjugate in $\mathrm{GL}(2,q)$ to a matrix
$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}. \tag{6.8}$$
To compute the centraliser of elements of this form we use the same approach as in the first two parts of this proof. We get that g is centralised by all diagonal matrices in $\mathrm{GL}(2,q)$. Surely, there are $(q-1)^2$ different diagonal matrices and we have $c_t(q) = (q-1)^2$.

   The conjugacy class is completely determined by the pair $\{a,b\}$. We can choose $a \in \mathbb{F}_q^*$ freely and then we choose $b \in \mathbb{F}_q^* \setminus \{a\}$. The order of those two elements does not matter and we obtain $k_t(q) = (q-1)(q-2)/2$.

□

Now, the next step is to investigate the set $\mathcal{O}(t)$ for each of the four types t. We want to determine all types $\bar{t}$ of matrices in $\mathcal{O}(t)$. Additionally, we need to compute the numbers $A(t,\bar{t})$. It is helpful to note that the following holds:
$$\sum_{\bar{t}\text{ type of }\mathcal{O}(t)} A(t,\bar{t}) = k_t. \tag{6.9}$$
The above Equation (6.9) is used as a simple check: If that equation does not hold, then at least one number $A(t,\bar{t})$ must be wrong.
The following Lemma 6.2 states the occurring types $\bar{t}$ and the corresponding numbers $A(t,\bar{t})$.

Lemma 6.2 *Let q be an arbitrary prime power. As usual, t denotes a type of matrices in $\mathrm{GL}(2,q)$ and $\bar{t}$ denotes a resulting type in $\mathcal{O}(t)$.*

1. *The type $t = ((2,(1)))$ yields $\bar{t}_1 = ((2,(1)),(1,(1,1)))$ and $\bar{t}_2 = ((1,(1,1)),(1,(1,1)))$. The corresponding numbers are*
$$A(t,\bar{t}_1) = q\cdot[2-(q,2)]/2 + [(q,2)-2]/2, \tag{6.10}$$
$$A(t,\bar{t}_2) = q^2/2 + q\cdot[(q,2)-3]/2 + [2-(q,2)]/2. \tag{6.11}$$
2. *The type $t = ((1,(2)))$ yields the types $\bar{t}_1 = ((1,(3,1)))$ and $\bar{t}_2 = ((1,(2,2)))$. The corresponding numbers are*
$$A(t,\bar{t}_1) = q\cdot[2-(q,2)] + [(q,2)-2], \tag{6.12}$$
$$A(t,\bar{t}_2) = q\cdot[(q,2)-1] - [(q,2)-1]. \tag{6.13}$$
3. *The type $t = ((1,(1,1)))$ yields only $\bar{t}_1 = ((1,(1,1,1,1)))$. The corresponding number is*
$$A(t,\bar{t}_1) = q-1. \tag{6.14}$$
4. *The type $t = ((1,(1)),(1,(1)))$ splits into the two types $\bar{t}_1 = ((1,(1)),(1,(1)),(1,(1,1)))$ and $\bar{t}_2 = ((1,(1,1)),(1,(1,1)))$. The corresponding numbers are*
$$A(t,\bar{t}_1) = q^2/2 + q\cdot[(q,2)-5]/2 + [4-(q,2)]/2, \tag{6.15}$$
$$A(t,\bar{t}_2) = q\cdot[2-(q,2)] + [(q,2)-2]. \tag{6.16}$$

Proof:

1. We have already discussed this type in Section 5.2, compare Example 12 on page 43. Equations (5.59) and (5.60) give the numbers $A(t,\bar{t})$ which we are looking for.

2. Let g be an element of type $t = ((1,(2)))$. Then, g is conjugate to the Jordan block $J_2(a)$ in $\mathrm{GL}(2,q)$ as well as in $\mathrm{GL}(2,\mathbb{E})$. Now, we are in the same situation as in the example on page 37 where we considered the Jordan decomposition of a Kronecker product of matrices. We have to distinguish between even and odd characteristic.

   In the case of odd characteristic (such a characteristic is not exceptional) we know that we can apply Theorem 5.6. We get that $J_2(a) \otimes J_2(a)$ is conjugate to a matrix $J_3(a^2) \oplus J_1(a^2)$. We obtain the type $\bar{t}_1 = ((1,(3,1)))$. In the case of an even characteristic we compute that $J_2(a) \otimes J_2(a)$ is conjugate to the matrix $J_2(a^2) \oplus J_2(a^2)$ which gives the type $\bar{t}_2 = ((1,(2,2)))$.

   In case of odd characteristic every matrix of type t will have type $\bar{t}_1$ in $\mathcal{O}(t)$, hence it is $A(t,\bar{t}_1) = k_t = q-1$ and $A(t,\bar{t}_2) = 0$. In case of even characteristic these two numbers are swapped yielding $A(t,\bar{t}_1) = 0$ and $A(t,\bar{t}_2) = k_t = q-1$. Combining these results to a PORC function (as in the example on page 43) we get the function as in this lemma.

3. Let g be an element of the type $t = ((1,(1,1)))$ such that g is conjugate to $J_1(a) \oplus J_1(a)$ in $\mathrm{GL}(2,q)$ as well as in $\mathrm{GL}(2,\mathbb{E})$. Thus $g \otimes g$ is conjugate to the 4-fold block-sum

$$J_1(a^2) \oplus J_1(a^2) \oplus J_1(a^2) \oplus J_1(a^2). \tag{6.17}$$

   This is independent from the characteristic. The resulting type is $\bar{t}_1 = ((1,(1,1,1,1)))$ and we get $A(t,\bar{t}_1) = k_t = q-1$.

4. At last, let g be an element of type $t = ((1,(1)),(1,(1)))$. Then, g is conjugate to $J_1(a) \oplus J_1(b)$ in $\mathrm{GL}(2,q) = \mathrm{GL}(2,\mathbb{E})$ with $a \neq b$ and $a,b \in \mathbb{F}_q$. Thus $g \otimes g$ is conjugate to

$$J_1(a^2) \oplus J_1(ab) \oplus J_1(ab) \oplus J_1(b^2). \tag{6.18}$$

   The type of this matrix depends on the coincidences among a^2, b^2, and ab. We directly see that the equations $a^2 = ab$ and $b^2 = ab$ cannot hold as this would imply $a = b$. Hence, the only possible coincidence is $a^2 = b^2$. In a field of even characteristic this coincidence cannot arise as it would imply $a = b$. But if the characteristic of the field is odd then this coincidence arises if and only if $b = -a$.

   The case $a^2 \neq b^2$ gives the type $\bar{t}_1 = ((1,(1,1)),(1,(1)),(1,(1)))$ and the other case with $a^2 = b^2$ gives $\bar{t}_2 = ((1,(1,1)),(1,(1,1)))$.

   We know that there are $(q-1)(q-2)/2$ different matrices of type t in $\mathrm{GL}(2,q)$. In even characteristic all of them yield type $\bar{t}_1$ and none of them yield type $\bar{t}_2$. However, in odd characteristic there are $(q-1)/2$ cases with $a = -b$ yielding $a^2 = b^2$. So, we have $A(t,\bar{t}_2) = (q-1)/2$. We get that the remaining cases will give type $\bar{t}_1$. The numbers are summarised in Table 6.1.

$t = ((1,(1)),(1,(1)))$	q even	q odd
$\bar{t}_1 = ((1,(1,1)),(1,(1)),(1,(1)))$	$(q-1)(q-2)/2$	$(q-1)(q-3)/2$
$\bar{t}_2 = ((1,(1,1)),(1,(1,1)))$	0	$(q-1)/2$

Table 6.1: The numbers $A(t,\bar{t}_i)$ for all types $\bar{t}_i$ arising from $t = ((1,(1)),(1,(1)))$.

When combining the results of Table 6.1 to a PORC function we get the function as in the lemma. □

So far, we have all types t of matrices in $\mathrm{GL}(2,q)$ and we have found, depending on t, all possible types $\bar{t}_i$ of matrices in $\mathcal{O}(t)$ together with their number of occurrences $A(t,\bar{t}_i)$. It remains to determine the fixed points $F_k(t)$. The result is given in Lemma 6.3. At this point we will omit a proof for this lemma. The computation was done using the algorithm of the `AutPGrp` package [11] for `GAP` [15].

Lemma 6.3 *Let q be an arbitrary prime power. Then, the number $F_k(t)$ of fixed points of matrices of type t acting on the k-dimensional subspaces of $T(r,q)$ is given in Table 6.2.*

type	$k=1$	$k=2$
$t=((2,(1)),(1,(1,1)))$	$q+1$	2
$t=((1,(1,1)),(1,(1,1)))$	$2(q+1)$	q^2+2q+3
$t=((1,(3,1)))$	$q+1$	$q+1$
$t=((1,(2,2)))$	$q+1$	q^2+q+1
$t=((1,(1,1,1,1)))$	q^3+q^2+q+1	$q^4+q^3+2q^2+q+1$
$t=((1,(1)),(1,(1)),(1,(1,1)))$	$q+3$	$2(q+2)$

Table 6.2: Number of fixed points when the group $\mathrm{GL}(2,q)$ acts via the Kronecker product on the set of all k-dimensional subspaces of the vector space $T(r,q)$.

As a last step, we need to combine the information provided in this section. As in Algorithm 1 we determine the PORC function via

$$N_{2+k,2}(q)=\sum_{t\in T}\frac{1}{c_t}\sum_{\bar{t}\in\mathcal{O}(t)}A(t,\bar{t})\cdot F_k(\bar{t}). \tag{6.19}$$

This yields the functions

$$N_{3,2}(q)=q+5-(q,2) \quad \text{and} \tag{6.20}$$
$$N_{4,2}(q)=3q+6-(q,2). \tag{6.21}$$

Thus, we have given an explicit proof for Theorem 1.5.

Chapter 7

Summary

7.1 Summary

In 1960 Higman [17, 18] worked on the determination of the number isomorphism types of p-groups of order p^n and introduced the term PORC function. Given an infinite subset S of $\mathbb{N}$, a function $f : S \to \mathbb{Q}$ is a polynomial on residue classes (PORC) if there exists a natural number $m \in \mathbb{N}$ and polynomials $g_0, \ldots, g_{m-1} \in \mathbb{Q}[x]$ such that $f(x) = g_a(x)$ holds for all $x \in S$ with $x \equiv a \bmod m$.
Higman formulated the well known PORC conjecture stating that the number of isomorphism types of p-groups of order p^n, considered as a function in p, is PORC. His conjecture has been proved true for $n \leq 7$, but it is still open for $n \geq 8$ [1, 2, 3, 4, 5, 7, 18, 20, 21, 22, 27, 29, 32].
In this thesis, nilpotent associative $\mathbb{F}_q$-algebras of dimension d, rank r and class 2 are considered. The number of isomorphism types of nilpotent associative $\mathbb{F}_q$-algebras of rank r, dimension d and class 2 is denoted by $N_{d,r}(q)$. The aims of this thesis can be summarised as follows:

- Investigate the properties of $N_{d,r}(q)$.
- Develop an algorithm to determine the numbers $N_{d,r}(q)$ for given d and r as a function in q.
- Explicitly compute functions $N_{d,r}(q)$ for small d and r.

Investigation. In Theorem 1.3 we show that $N_{d,r}(q)$ is PORC, considered as a function in q. Our proof is independent of Higmans's results. Then, Theorem 1.4 establishes further properties of $N_{d,r}(q)$:

- $N_{d,r}(q) = 0$ if $d \notin \{r+1, \ldots, r+r^2\}$,
- $N_{r+r^2,r}(q) = 1$, and
- $N_{r+k,r}(q) = N_{r+r^2-k,r}(q)$ for all $k \in \{1, \ldots, r^2-1\}$.

It follows that for a given rank r it is only $N_{d,r}(q) \neq 0$ for dimensions d with $r+1 \leq d \leq r+r^2$. Hence, for a given rank r only finitely many functions $N_{d,r}(q)$ need to be determined.

Algorithm. The first step of the algorithm is to translate the counting of isomorphism types of nilpotent associative $\mathbb{F}_q$-algebras of class 2, dimension d, and rank r to an orbit-stabiliser-problem. It is shown that $N_{r+k,r}(q)$ coincides with the number of orbits of $\mathrm{GL}(r,q)$ acting on the set $\mathcal{U}_k(r,q)$ which is the set of k-dimensional subspaces of the $\mathbb{F}_q$-vector space $T(r,q) = \mathbb{F}_q^r \otimes_{\mathbb{F}_q} \mathbb{F}_q^r$ (Theorem 4.6). Using the counting lemma of Burnside, Cauchy, and Frobenius [31, Theorem 3.22], Corollary 4.8 establishes a formula for $N_{r+k,r}(q)$ (compare Equation (7.1)). There, $\mathrm{Fix}_g(\mathcal{U}_k(r,q))$ denotes the number of fixed points among $\mathcal{U}_k(r,q)$ under the action of $g \in \mathrm{GL}(r,q)$.

$$N_{r+k,r}(q) = \frac{1}{|\mathrm{GL}(r,q)|} \sum_{g \in \mathrm{GL}(r,q)} \mathrm{Fix}_g(\mathcal{U}_k(r,q)). \tag{7.1}$$

Then, an improved version of Equation (7.1) is formulated such that the summation is independent of the field $\mathbb{F}_q$. For this purpose, types of matrices are introduced. These are a generalisation of the rational canonical form of a matrix. The types of matrices in $\mathrm{GL}(r,q)$ are independent of the field $\mathbb{F}_q$, they only depend on r. Matrices of the same type share some properties. For instance, the order of the centraliser c_t of matrices of type t in $\mathrm{GL}(r,q)$ is independent of the matrix itself. Further, c_t can be considered as a polynomial in q for a fixed t (Lemma 3.10). When a matrix $g \in \mathrm{GL}(r,q)$ of type t acts on the set of k-dimensional subspaces of $\mathbb{F}_q^r$, the number $F_k(t)$ of fixed points is independent of g. $F_k(t)$ only depends on the type t and for a fixed type t it can be considered as a polynomial in q (Lemma 3.11 and [10, Section 4.2]).
Since $\mathrm{GL}(r,q)$ acts on the tensor product space $T(r,q)$, the set

$$\mathcal{O}(t) = \{g \otimes g \mid g \in \mathrm{GL}(r,q) \text{ has type } t\} \subseteq \mathrm{GL}(r^2,q) \tag{7.2}$$

has to be considered. In $\mathcal{O}(t)$ matrices of different types in $\mathrm{GL}(r^2,q)$ can occur. Let $\bar{t}$ be a type of a matrix in $\mathcal{O}(t)$ and let $A(t,\bar{t})$ denote the number of different conjugacy classes of matrices of type $\bar{t}$ in $\mathcal{O}(t)$. In Chapter 5 an algorithm is introduced which determines all types $\bar{t}$ in $\mathcal{O}(t)$ and the corresponding numbers $A(t,\bar{t})$ for a given type t. Further, it is shown that the number $A(t,\bar{t})$, considered as a function in q, is PORC (Corollary 5.12). It remains to compute the number $F_k(\bar{t})$ of fixed points of a matrix in $\mathcal{O}(t)$ of type $\bar{t}$ under the action on $\mathcal{U}_k(r,q)$. The number $F_k(\bar{t})$ only depends on $\bar{t}$, not on the matrix itself. Furthermore, for a fixed $\bar{t}$ and considered as a function in q, $F_k(\bar{t})$ is a polynomial in q (Lemma 3.11 and [10, Section 4.2]).
Combining these results one gets a formulation of Equation (7.1) such that the summation is independent of the field $\mathbb{F}_q$ (Corollary 4.9). The set of all types of matrices in $\mathrm{GL}(r,q)$ is denoted by T.

$$N_{r+k,r}(q) = \sum_{t \in T} \frac{1}{c_t} \sum_{\bar{t} \in \mathcal{O}(t)} A(t,\bar{t}) \cdot F_k(\bar{t}). \tag{7.3}$$

It follows from Lemma 3.10 that c_t divides $\sum_{\bar{t} \in \mathcal{O}(t)} A(t,\bar{t}) \cdot F_k(\bar{t})$. Since c_t and $F_k(\bar{t})$ can be considered as polynomials in q and since $A(t,\bar{t})$ can be considered as a PORC function in q, $N_{r+k,r}(q)$ is PORC and Equation (7.3) can be translated to an algorithm (see Algorithm 1).
The runtime of the determination of $A(t,\bar{t})$ is considered in Section 5.3. To determine $A(t,\bar{t})$ systems of equations must be considered and the number of such systems is given by $\mathcal{O}(2^{r^4})$.
Using a PC with a 3,40 GHz Intel processor and 32 GB RAM, our algorithm terminates for $r \in \{1,2,3,4\}$ in a few seconds, however, for $r = 5$ it already takes about 10 hours. The main bottleneck of our implementation is the determination of the different types $\bar{t}$ of matrices in $\mathcal{O}(t)$ and their corresponding numbers $A(t,\bar{t})$.

Explicit functions. We use our algorithm to determine the functions $N_{d,r}(q)$ for rank r up to 5 and for all d. The functions $N_{d,r}(q)$ with $1 \leq r \leq 5$ are exhibited in Appendix A. Based on our computations we conjecture the following properties of the degrees of the functions $N_{d,r}(q)$. The conjecture holds for $r \in \{1, 2, 3, 4, 5\}$.

- $N_{r+1,r}(q) = N_{r+r^2-1,r}(q)$ is homogeneous of degree $\lfloor r/2 \rfloor$.
- $N_{r+k,r}(q)$ is homogeneous of degree $1 + r^2k - (r^2 + k^2)$ for $k \in \{2, \ldots, r^2 - 2\}$.

Final comments. Our algorithm can be extended by the implementation of further group actions. For instance, using the wedge product space $\mathbb{F}_q^r \wedge \mathbb{F}_q^r$ (instead of the Tensor product space) one can determine the number of isomorphism types of class 2 Lie algebras with r generators. When the group $\mathrm{GL}(r, p)$ acts on the vector space $(\mathbb{F}_p^r \wedge \mathbb{F}_p^r) \otimes \mathbb{F}_p^r$ one obtains the number of isomorphism types of class 2 p-groups with r generators.

7.2 Zusammenfassung

Higman [17, 18] befasste sich 1960 mit der Bestimmung der Anzahl der Isomorphietypen von p-Gruppen der Ordnung p^n. In diesem Zusammenhang führte er den Begriff PORC-Funktion ein. Sei S eine unendliche Teilmenge der natürlichen Zahlen $\mathbb{N}$. Eine Funktion $f : S \to \mathbb{Q}$ heißt PORC (polynomial on residue classes, zu deutsch etwa *Polynom auf Restklassen*), falls es eine natürliche Zahl $m \in \mathbb{N}$ und Polynome $g_0, \ldots, g_{m-1} \in \mathbb{Q}[x]$ gibt, für die gilt: Es ist $f(x) = g_a(x)$ für alle $x \in S$ mit $x \equiv a \bmod m$.
Higman formulierte seine wohl bekannte PORC-Vermutung, welche besagt, dass die Anzahl der Isomorphietypen von p-Gruppen der Ordnung p^n aufgefasst als Funktion in p PORC ist. Higmans Vermutung wurde für $n \leq 7$ bewiesen, jedoch ist dieses Problem für $n \geq 8$ immer noch ungelöst [1, 2, 3, 4, 5, 7, 18, 20, 21, 22, 27, 29, 32].
Diese Arbeit befasst sich mit nilpotenten assoziativen $\mathbb{F}_q$-Algebren der Klasse 2, Dimension d und vom Rang r. Die Anzahl der Isomorphietypen von nilpotenten assoziativen $\mathbb{F}_q$-Algebren der Klasse 2, Dimension d und vom Rang r wird mit $N_{d,r}(q)$ bezeichnet. Die Ziele dieser Arbeit lassen sich wie folgt zusammenfassen:

- Untersuchung der Eigenschaften von $N_{d,r}(q)$.
- Entwicklung eines Algorithmus, mit welchem $N_{d,r}(q)$ für gegebenes d und r als Funktion in q bestimmt werden kann.
- Explizites Berechnen der Funktionen $N_{d,r}(q)$ für kleine Werte d und r.

Untersuchung der Eigenschaften. In Theorem 1.3 wird gezeigt, dass $N_{d,r}(q)$ aufgefasst als Funktion in q PORC ist. Dabei ist der Beweis in dieser Arbeit unabhängig von Higmans Resultaten. Mit Theorem 1.4 folgen weitere Eigenschaften von $N_{d,r}(q)$:

- $N_{d,r}(q) = 0$, falls $d \notin \{r+1, \ldots, r+r^2\}$,
- $N_{r+r^2,r}(q) = 1$ und
- $N_{r+k,r}(q) = N_{r+r^2-k,r}(q)$ für alle $k \in \{1, \ldots, r^2-1\}$.

Für festen Rang r ergibt sich, dass $N_{d,r}(q) \neq 0$ nur für Dimensionen d mit $r+1 \leq d \leq r+r^2$ gilt. Damit müssen für einen festen Rang r nur endlich viele Funktionen $N_{d,r}(q)$ bestimmt werden.

Algorithmus. Der erste Schritt bei der Entwicklung des Algorithmus ist der folgende: Man übersetzt das Zählen der Isomorphietypen von nilpotenten assoziativen $\mathbb{F}_q$-Algebren mit Klasse 2, Dimension d und Rang r in einen Bahn-Stabilisator-Algorithmus. Die Zahl $N_{d,r}(q)$ stimmt mit der Anzahl der Bahnen überein, wenn die Gruppe $\mathrm{GL}(r,q)$ auf der Menge $\mathcal{U}_k(r,q)$ operiert. Dabei bezeichnet $\mathcal{U}_k(r,q)$ die Menge der k-dimensionalen $\mathbb{F}_q$-Unterräume von $T(r,q) = \mathbb{F}_q^r \otimes_{\mathbb{F}_q} \mathbb{F}_q^r$ (Theorem 4.6). Mit Hilfe des Lemmas von Burnside, Cauchy und Frobenius [31, Theorem 3.22] folgt Korollar 4.8, welches eine Formel zur Bestimmung von $N_{d,r}(q)$ angibt, siehe Gleichung (7.4). In dieser Gleichung bezeichnet $\mathrm{Fix}_g(\mathcal{U}_k(r,q))$ die Anzahl der Fixpunkte der Wirkung der Matrix $g \in \mathrm{GL}(r,q)$ auf der Menge $\mathcal{U}_k(r,q)$.

$$N_{r+k,r}(q) = \frac{1}{|\mathrm{GL}(r,q)|} \sum_{g \in \mathrm{GL}(r,q)} \mathrm{Fix}_g(\mathcal{U}_k(r,q)). \tag{7.4}$$

Als Nächstes wird eine Verbesserung von Gleichung (7.4) formuliert, sodass die Summation nicht mehr vom Körper $\mathbb{F}_q$ abhängt. Dazu wird der Typ einer Matrix eingeführt. Der Typ einer Matrix ist eine Verallgemeinerung ihrer rationalen Normalform. Die Typen der Matrizen aus $\mathrm{GL}(r,q)$ hängen nur von r ab, jedoch nicht vom Körper $\mathbb{F}_q$. Matrizen desselben Typs haben gemeinsame Eigenschaften. Unter Anderem ist die Ordnung c_t des Zentralisators einer Matrix vom Typ t in $\mathrm{GL}(r,q)$ unabhängig von der Matrix. Für einen festen Typ t kann die Anzahl c_t als Polynom in q aufgefasst werden (Lemma 3.10). Wenn eine Matrix vom Typ t auf der Menge der k-dimensionalen Unterräume des Vektorraums $\mathbb{F}_q^r$ operiert, dann ist die Anzahl $F_k(t)$ der Fixpunkte ebenfalls unabhängig von der gewählten Matrix. Die Zahl $F_k(t)$ hängt nur vom Typ t ab und kann als Polynom in q aufgefasst werden (Lemma 3.11 und [10, Section 4.2]). Da die Gruppe $\mathrm{GL}(r,q)$ auf dem Tensorprodukt-Raum $T(r,q)$ operiert, muss die nachstehende Menge $\mathcal{O}(t)$ betrachtet werden:

$$\mathcal{O}(t) = \{g \otimes g \mid g \in \mathrm{GL}(r,q) \text{ ist vom Typ } t\} \subseteq \mathrm{GL}(r^2,q). \tag{7.5}$$

Die Matrizen in $\mathcal{O}(t)$ können verschiedene Typen in $\mathrm{GL}(r^2,q)$ haben. Sei dazu $\bar{t}$ ein Typ einer Matrix aus $\mathcal{O}(t)$ und sei $A(t,\bar{t})$ die Anzahl der verschiedenen Konjugationsklassen von Matrizen vom Typ $\bar{t}$ in $\mathcal{O}(t)$. In Kapitel 5 wird dazu ein Algorithmus vorgestellt, welcher sowohl alle Typen $\bar{t}$ in $\mathcal{O}(t)$ als auch die Zahlen $A(t,\bar{t})$ bestimmt. Es wird des Weiteren gezeigt, dass $A(t,\bar{t})$ als Funktion in q PORC ist (Korollar 5.12). Es verbleibt die Anzahl der Fixpunkte $F_k(\bar{t})$ zu bestimmen, wenn eine Matrix aus $\mathcal{O}(t)$ vom Typ $\bar{t}$ auf $\mathcal{U}_k(r,q)$ operiert. Die Zahl $F_k(\bar{t})$ hängt nur von $\bar{t}$ an, nicht von der gewählten Matrix. Die Zahl $F_k(\bar{t})$ kann als Funktion in q aufgefasst werden und ist dann ein Polynom in q (Lemma 3.11 und [10, Section 4.2]).
Kombiniert man all diese Ergebnisse, dann erhält man eine Verbesserung von Formel (7.4), sodass die Summation nicht mehr vom Körper $\mathbb{F}_q$ abhängt. Dabei ist T die Menge aller Typen von Matrizen in $\mathrm{GL}(r,q)$.

$$N_{r+k,r}(q) = \sum_{t \in T} \frac{1}{c_t} \sum_{\bar{t} \in \mathcal{O}(t)} A(t,\bar{t}) \cdot F_k(\bar{t}). \tag{7.6}$$

Es folgt aus Lemma 3.10, dass c_t den Term $\sum_{\bar{t} \in \mathcal{O}(t)} A(t,\bar{t}) \cdot F_k(\bar{t})$ teilt. Weil c_t und $F_k(\bar{t})$ als Polynome in q aufgefasst werden können und weil $A(t,\bar{t})$ als PORC-Funktion aufgefasst werden kann, folgt, dass

$N_{r+k,r}(q)$ eine PORC-Funktion ist. Gleichung (7.6) lässt sich direkt in einen Algorithmus umformen (vergleiche Algorithmus 7.3).
Die Laufzeit der Bestimmung von $A(t, \bar{t})$ wird in Abschnitt 5.3 untersucht. Zur Bestimmung von $A(t, \bar{t})$ müssen Gleichungssysteme gelöst werden und die Anzahl der Gleichungssysteme kann mit $\mathcal{O}(2^{r^4})$ abgeschätzt werden.
Der in dieser Arbeit vorgestellte Algorithmus terminiert auf einem PC mit einem 3,40 GHz Intel Prozessor und einem Arbeitsspeicher von 32 GB für $r \in \{1, 2, 3, 4\}$ innerhalb weniger Sekunden. Für $r = 5$ benötigt er bereits zehn Stunden. Dabei ist der zeitaufwendigste Teil des Algorithmus die Bestimmung der Typen $\bar{t}$ in $\mathcal{O}(t)$ sowie der dazu gehörigen Funktionen $A(t, \bar{t})$.

Funktionen. Mit Hilfe des Algorithmus, der in dieser Arbeit vorgestellt wurde, werden die Funktionen $N_{d,r}(q)$ für $1 \leq r \leq 5$ und jedes mögliche d berechnet. Diese Funktionen sind in Anhang A zu finden. Basierend auf diesen Berechnungen lassen sich nachstehende Vermutungen formulieren. Diese Vermutungen gelten für $r \in \{1, 2, 3, 4, 5\}$.

- $N_{r+1,r}(q) = N_{r+r^2-1,r}(q)$ ist homogen vom Grad $\lfloor r/2 \rfloor$. Das bedeutet, dass alle auftretenden Polynome in q den Grad $\lfloor r/2 \rfloor$ haben.
- $N_{r+k,r}(q)$ ist homogen vom Grad $1 + r^2k - (r^2 + k^2)$ für $k \in \{2, \ldots, r^2 - 2\}$.

Abschließende Bemerkungen. Durch die Implementation weiterer Gruppenoperationen lässt sich der Algorithmus erweitern. Nutzt man beispielsweise den Wedgeprodukt-Raum $\mathbb{F}_q^r \wedge \mathbb{F}_q^r$ (anstelle des Tensorprodukt-Raums), dann kann man mit Hilfe des Algorithmus die Anzahl der Isomorphietypen von Lie-Algebren von Klasse 2 mit r Erzeugern zählen. Nutzt man hingegen die Operation auf dem Vektorraum $(\mathbb{F}_p^r \wedge \mathbb{F}_p^r) \otimes \mathbb{F}_p^r$, dann erhält man die Anzahl der Isomorphietypen von Klasse 2 p-Gruppen mit r Erzeugern.

Chapter A

PORC functions

A.1 Notes on Appendix A

Here, we expose all PORC functions $N_{d,r}(q)$ for $r \in \{2,3,4,5\}$ and all d. We recall the rules which we apply to PORC functions.

1. Instead of writing $\gcd(a,b)$ it is simply written (a,b).
2. Square brackets [] are used for grouping terms together.

We use the symmetry in the degrees d of the functions $N_{d,r}(q)$ which is given by $N_{d,r}(q) = N_{r+r^2-d,r}(q)$ and, therefore, only expose the functions $N_{d,r}(q)$ for $d \in \{r+1,\ldots,r+\lceil \frac{r^2}{2} \rceil\}$ for fixed r (compare Theorem 1.4).

A.2 Rank two

A.2.1 Dimension $d = 3$

$N_{3,2}(q) = q - (q,2) + 5$

A.2.2 Dimension $d = 4$

$N_{4,2}(q) = 3 \cdot q - (q,2) + 6$

A.3 Rank three

A.3.1 Dimension $d = 4$

$$N_{4,3}(q) = 2 \cdot q + [-2 \cdot (q,2) + 11]$$

A.3.2 Dimension $d = 5$

$$\begin{aligned} N_{5,3}(q) = {} & q^6 + q^5 + 3 \cdot q^4 + 6 \cdot q^3 + [-2 \cdot (q,2) + 18] \cdot q^2 + [-7 \cdot (q,2) + (q-1,3) + 38] \cdot q \\ & + [-10 \cdot (q,2) - 1/2 \cdot (q,3) + 89/2] \end{aligned}$$

A.3.3 Dimension $d = 6$

$$\begin{aligned} N_{6,3}(q) = {} & q^{10} + q^9 + 3 \cdot q^8 + 5 \cdot q^7 + 8 \cdot q^6 + 13 \cdot q^5 + [-2 \cdot (q,2) + 29] \cdot q^4 \\ & + [-6 \cdot (q,2) + 48] \cdot q^3 + [-15 \cdot (q,2) - 1/2 \cdot (q,3) + 2 \cdot (q-1,3) + (q-1,4) + 177/2] \cdot q^2 \\ & + [-22 \cdot (q,2) - 1/2 \cdot (q,3) + 4 \cdot (q-1,3) + 2 \cdot (q-1,4) + 223/2] \cdot q \\ & + [-20 \cdot (q,2) - 1/2 \cdot (q,3) + (q-1,3) + (q-1,4) + 173/2] \end{aligned}$$

A.3.4 Dimension $d = 7$

$$\begin{aligned} N_{7,3}(q) = {} & q^{12} + q^{11} + 3 \cdot q^{10} + 5 \cdot q^9 + 9 \cdot q^8 + 13 \cdot q^7 + 22 \cdot q^6 + [-(q,2) + 34] \cdot q^5 \\ & + [-7 \cdot (q,2) + 65] \cdot q^4 + [-17 \cdot (q,2) + 1/2 \cdot (q,3) + 2 \cdot (q-1,3) + 215/2] \cdot q^3 \\ & + [-32 \cdot (q,2) - (q,3) + 5 \cdot (q-1,3) + 2 \cdot (q-1,4) + 166] \cdot q^2 \\ & + [-39 \cdot (q,2) - 1/2 \cdot (q,3) + 7 \cdot (q-1,3) + 4 \cdot (q-1,4) + 361/2] \cdot q \\ & + [-28 \cdot (q,2) - (q,3) + 2 \cdot (q-1,3) + 2 \cdot (q-1,4) + 121] \end{aligned}$$

A.4 Rank four

A.4.1 Dimension $d = 5$

$$N_{5,4}(q) = q^2 + [-(q,2) + 7] \cdot q + [-6 \cdot (q,2) + 25]$$

A.4.2 Dimension $d = 6$

$$\begin{aligned}
N_{6,4}(q) = {}& q^{13} + q^{12} + 3 \cdot q^{11} + 4 \cdot q^{10} + 8 \cdot q^9 + 10 \cdot q^8 + [-(q,2) + 21] \cdot q^7 \\
&+ [-3 \cdot (q,2) + 32] \cdot q^6 + [-8 \cdot (q,2) + 58] \cdot q^5 \\
&+ [-16 \cdot (q,2) + 1/2 \cdot (q,3) + (q-1,3) + 181/2] \cdot q^4 \\
&+ [-33 \cdot (q,2) + 3 \cdot (q-1,3) + 2 \cdot (q-1,4) + 156] \cdot q^3 \\
&+ [-(q,2) \cdot (q-1,3) - 55 \cdot (q,2) + (q,3) + 9 \cdot (q-1,3) + 4 \cdot (q-1,4) + 229] \cdot q^2 \\
&+ [-2 \cdot (q,2) \cdot (q-1,3) - 77 \cdot (q,2) - 1/2 \cdot (q,3) + 13 \cdot (q-1,3) + 6 \cdot (q-1,4) + 583/2] \cdot q \\
&+ [-(q,2) \cdot (q-1,3) - 59 \cdot (q,2) - 1/2 \cdot (q,3) + 6 \cdot (q-1,3) + 4 \cdot (q-1,4) + 415/2]
\end{aligned}$$

A.4.3 Dimension $d = 7$

$$\begin{aligned}
N_{7,4}(q) = {}& q^{24} + q^{23} + 3 \cdot q^{22} + 5 \cdot q^{21} + 9 \cdot q^{20} + 13 \cdot q^{19} + 22 \cdot q^{18} + 30 \cdot q^{17} + 45 \cdot q^{16} \\
&+ 61 \cdot q^{15} + 85 \cdot q^{14} + 111 \cdot q^{13} + [-2 \cdot (q,2) + 157] \cdot q^{12} + [-5 \cdot (q,2) + 208] \cdot q^{11} \\
&+ [-16 \cdot (q,2) + 296] \cdot q^{10} + [-33 \cdot (q,2) + 405] \cdot q^9 + [-71 \cdot (q,2) + 580] \cdot q^8 \\
&+ [-120 \cdot (q,2) + (q,3) + 3 \cdot (q-1,3) + 786] \cdot q^7 \\
&+ [-207 \cdot (q,2) + 1/2 \cdot (q,3) + 10 \cdot (q-1,3) + (q-1,4) + 2187/2] \cdot q^6 \\
&+ [-308 \cdot (q,2) + (q,3) + 28 \cdot (q-1,3) + 8 \cdot (q-1,4) + 1426] \cdot q^5 \\
&+ [-455 \cdot (q,2) + 57 \cdot (q-1,3) + 26 \cdot (q-1,4) + (q-1,5) + 1854] \cdot q^4 \\
&+ [-6 \cdot (q,2) \cdot (q-1,3) - 582 \cdot (q,2) + 1/2 \cdot (q,3) + 111 \cdot (q-1,3) + 59 \cdot (q-1,4) + 5 \cdot (q-1,5) \\
&\quad +1/3 \cdot (q-1,7) + 1/3 \cdot (q-2,7) + 1/3 \cdot (q-4,7) + 4397/2] \cdot q^3 \\
&+ [-20 \cdot (q,2) \cdot (q-1,3) - 652 \cdot (q,2) - 2 \cdot (q,3) + 169 \cdot (q-1,3) + 92 \cdot (q-1,4) + 9 \cdot (q-1,5) \\
&\quad +(q-1,7) + (q-1,8) + 2346] \cdot q^2 \\
&+ [-25 \cdot (q,2) \cdot (q-1,3) - 545 \cdot (q,2) - 7/2 \cdot (q,3) + 153 \cdot (q-1,3) + 87 \cdot (q-1,4) + 7 \cdot (q-1,5) \\
&\quad +(q-1,7) + 2 \cdot (q-1,8) + 3793/2] \cdot q \\
&+ [-9 \cdot (q,2) \cdot (q-1,3) - 264 \cdot (q,2) - 7/2 \cdot (q,3) + 51 \cdot (q-1,3) + 35 \cdot (q-1,4) + 2 \cdot (q-1,5) \\
&\quad +1/3 \cdot (q-1,7) + 1/3 \cdot (q-2,7) + 1/3 \cdot (q-4,7) + (q-1,8) + 1765/2]
\end{aligned}$$

A.4.4 Dimension $d = 8$

$$\begin{aligned}
N_{8,4}(q) = {}& q^{33} + q^{32} + 3 \cdot q^{31} + 5 \cdot q^{30} + 10 \cdot q^{29} + 14 \cdot q^{28} + 25 \cdot q^{27} + 35 \cdot q^{26} + 55 \cdot q^{25} \\
&+ 75 \cdot q^{24} + 110 \cdot q^{23} + 146 \cdot q^{22} + 205 \cdot q^{21} + 264 \cdot q^{20} + 355 \cdot q^{19} \\
&+ 450 \cdot q^{18} + 587 \cdot q^{17} + [-2 \cdot (q,2) + 732] \cdot q^{16} + [-9 \cdot (q,2) + 951] \cdot q^{15} \\
&+ [-24 \cdot (q,2) + 1191] \cdot q^{14} + [-59 \cdot (q,2) + 1553] \cdot q^{13} + [-120 \cdot (q,2) + 1981] \cdot q^{12} \\
&+ [-231 \cdot (q,2) + 2605] \cdot q^{11} + [-395 \cdot (q,2) + (q,3) + 3 \cdot (q-1,3) + 3348] \cdot q^{10} \\
&+ [-652 \cdot (q,2) + (q,3) + 12 \cdot (q-1,3) + 4377] \cdot q^9 \\
&+ [-988 \cdot (q,2) + 3 \cdot (q,3) + 40 \cdot (q-1,3) + 2 \cdot (q-1,4) + 5544] \cdot q^8 \\
&+ [-1449 \cdot (q,2) + 3/2 \cdot (q,3) + 96 \cdot (q-1,3) + 16 \cdot (q-1,4) + 14049/2] \cdot q^7 \\
&+ [-1987 \cdot (q,2) + 7/2 \cdot (q,3) + 202 \cdot (q-1,3) \\
&+ [+57 \cdot (q-1,4) + 1/4 \cdot (q,5) + (q-1,5) + 34237/4] \cdot q^6 \\
&+ [-(q,2) \cdot (q-1,3) - 2604 \cdot (q,2) - 5/2 \cdot (q,3) + 358 \cdot (q-1,3) + 153 \cdot (q-1,4) - 1/4 \cdot (q,5) \\
&\quad +7 \cdot (q-1,5) + 40695/4] \cdot q^5 \\
&+ [-18 \cdot (q,2) \cdot (q-1,3) - 3124 \cdot (q,2) + 3/2 \cdot (q,3) + 599 \cdot (q-1,3) + 311 \cdot (q-1,4) + 1/4 \cdot (q,5)
\end{aligned}$$

$$
\begin{aligned}
&\quad +27\cdot(q-1,5)+(q-1,7)+45269/4]\cdot q^4\\
&+[-77\cdot(q,2)\cdot(q-1,3)-3365\cdot(q,2)-6\cdot(q,3)+904\cdot(q-1,3)+504\cdot(q-1,4)-1/2\cdot(q,5)\\
&\quad +60\cdot(q-1,5)-1/2\cdot(q-2,5)-1/2\cdot(q-3,5)+16/3\cdot(q-1,7)+1/3\cdot(q-2,7)\\
&\quad +1/3\cdot(q-4,7)+4\cdot(q-1,8)+23263/2]\cdot q^3\\
&+[-148\cdot(q,2)\cdot(q-1,3)-3025\cdot(q,2)-9\cdot(q,3)+1064\cdot(q-1,3)+600\cdot(q-1,4)\\
&\quad +1/2\cdot(q,5)+79\cdot(q-1,5)+1/2\cdot(q-2,5)+1/2\cdot(q-3,5)+10\cdot(q-1,7)\\
&\quad +14\cdot(q-1,8)+20417/2]\cdot q^2\\
&+[-132\cdot(q,2)\cdot(q-1,3)-2017\cdot(q,2)-23/2\cdot(q,3)+753\cdot(q-1,3)+441\cdot(q-1,4)\\
&\quad -1/4\cdot(q,5)+52\cdot(q-1,5)+8\cdot(q-1,7)+16\cdot(q-1,8)+26863/4]\cdot q\\
&+[-43\cdot(q,2)\cdot(q-1,3)-762\cdot(q,2)-13/2\cdot(q,3)+223\cdot(q-1,3)+144\cdot(q-1,4)\\
&\quad +1/4\cdot(q,5)+14\cdot(q-1,5)+7/3\cdot(q-1,7)\\
&\quad +1/3\cdot(q-2,7)+1/3\cdot(q-4,7)+6\cdot(q-1,8)+9921/4]
\end{aligned}
$$

A.4.5 Dimension $d = 9$

$$
\begin{aligned}
N_{9,4}(q) = {}& q^{40}+q^{39}+3\cdot q^{38}+5\cdot q^{37}+10\cdot q^{36}+15\cdot q^{35}+26\cdot q^{34}+38\cdot q^{33}+60\cdot q^{32}\\
&+85\cdot q^{31}+125\cdot q^{30}+172\cdot q^{29}+242\cdot q^{28}+323\cdot q^{27}+437\cdot q^{26}\\
&+570\cdot q^{25}+747\cdot q^{24}+952\cdot q^{23}+1216\cdot q^{22}+1518\cdot q^{21}\\
&+[-(q,2)+1901]\cdot q^{20}+[-3\cdot(q,2)+2338]\cdot q^{19}+[-14\cdot(q,2)+2895]\cdot q^{18}\\
&+[-36\cdot(q,2)+3545]\cdot q^{17}+[-94\cdot(q,2)+4398]\cdot q^{16}+[-190\cdot(q,2)+5419]\cdot q^{15}\\
&+[-376\cdot(q,2)+6776]\cdot q^{14}+[-657\cdot(q,2)+1/2\cdot(q,3)+(q-1,3)+16867/2]\cdot q^{13}\\
&+[-1114\cdot(q,2)+(q,3)+5\cdot(q-1,3)+10612]\cdot q^{12}\\
&+[-1737\cdot(q,2)+7/2\cdot(q,3)+22\cdot(q-1,3)+26421/2]\cdot q^{11}\\
&+[-2633\cdot(q,2)+4\cdot(q,3)+64\cdot(q-1,3)+(q-1,4)+16491]\cdot q^{10}\\
&+[-3754\cdot(q,2)+7\cdot(q,3)+165\cdot(q-1,3)+13\cdot(q-1,4)+20224]\cdot q^{9}\\
&+[-5202\cdot(q,2)+9/2\cdot(q,3)+354\cdot(q-1,3)+58\cdot(q-1,4)+49167/2]\cdot q^{8}\\
&+[-6830\cdot(q,2)+6\cdot(q,3)+680\cdot(q-1,3)+184\cdot(q-1,4)+1/2\cdot(q,5)+3\cdot(q-1,5)+58123/2]\cdot q^{7}\\
&+[-2\cdot(q,2)\cdot(q-1,3)-8634\cdot(q,2)-(q,3)+1155\cdot(q-1,3)+446\cdot(q-1,4)-1/4\cdot(q,5)\\
&\quad +17\cdot(q-1,5)-1/2\cdot(q-2,5)-1/2\cdot(q-3,5)+134033/4]\cdot q^{6}\\
&+[-31\cdot(q,2)\cdot(q-1,3)-10179\cdot(q,2)+1/2\cdot(q,3)+1829\cdot(q-1,3)+894\cdot(q-1,4)\\
&\quad +65\cdot(q-1,5)+(q-1,7)+73427/2]\cdot q^{5}\\
&+[-145\cdot(q,2)\cdot(q-1,3)-11103\cdot(q,2)-15/2\cdot(q,3)+2701\cdot(q-1,3)+1486\cdot(q-1,4)\\
&\quad +165\cdot(q-1,5)+9\cdot(q-1,7)+6\cdot(q-1,8)+75645/2]\cdot q^{4}\\
&+[-380\cdot(q,2)\cdot(q-1,3)-10700\cdot(q,2)-23/2\cdot(q,3)+3575\cdot(q-1,3)+2014\cdot(q-1,4)\\
&\quad +286\cdot(q-1,5)+29\cdot(q-1,7)+32\cdot(q-1,8)+70449/2]\cdot q^{3}\\
&+[-547\cdot(q,2)\cdot(q-1,3)-8576\cdot(q,2)-28\cdot(q,3)+3594\cdot(q-1,3)+2038\cdot(q-1,4)\\
&\quad +1/2\cdot(q,5)+312\cdot(q-1,5)+1/2\cdot(q-2,5)+1/2\cdot(q-3,5)+43\cdot(q-1,7)+65\cdot(q-1,8)\\
&\quad +55701/2]\cdot q^{2}\\
&+[-413\cdot(q,2)\cdot(q-1,3)-5027\cdot(q,2)-19\cdot(q,3)+2242\cdot(q-1,3)+1304\cdot(q-1,4)\\
&\quad +186\cdot(q-1,5)+30\cdot(q-1,7)+58\cdot(q-1,8)+16240]\cdot q\\
&+[-120\cdot(q,2)\cdot(q-1,3)-1649\cdot(q,2)-14\cdot(q,3)+601\cdot(q-1,3)+380\cdot(q-1,4)\\
&\quad +1/4\cdot(q,5)+46\cdot(q-1,5)+8\cdot(q-1,7)+19\cdot(q-1,8)+21107/4]
\end{aligned}
$$

A.4.6 Dimension $d = 10$

$$
\begin{aligned}
N_{10,4}(q) = {} & q^{45} + q^{44} + 3 \cdot q^{43} + 5 \cdot q^{42} + 10 \cdot q^{41} + 15 \cdot q^{40} + 27 \cdot q^{39} + 39 \cdot q^{38} + 63 \cdot q^{37} \\
& + 90 \cdot q^{36} + 135 \cdot q^{35} + 186 \cdot q^{34} + 268 \cdot q^{33} + 359 \cdot q^{32} + 495 \cdot q^{31} \\
& + 650 \cdot q^{30} + 867 \cdot q^{29} + 1112 \cdot q^{28} + 1446 \cdot q^{27} + 1817 \cdot q^{26} + 2306 \cdot q^{25} \\
& + 2849 \cdot q^{24} + 3542 \cdot q^{23} + [-(q,2) + 4306] \cdot q^{22} + [-6 \cdot (q,2) + 5277] \cdot q^{21} \\
& + [-20 \cdot (q,2) + 6351] \cdot q^{20} + [-58 \cdot (q,2) + 7723] \cdot q^{19} + [-136 \cdot (q,2) + 9290] \cdot q^{18} \\
& + [-293 \cdot (q,2) + 11320] \cdot q^{17} + [-558 \cdot (q,2) + 13702] \cdot q^{16} \\
& + [-1011 \cdot (q,2) + 16823] \cdot q^{15} + [-1685 \cdot (q,2) + (q,3) + 3 \cdot (q-1,3) + 20523] \cdot q^{14} \\
& + [-2703 \cdot (q,2) + 3/2 \cdot (q,3) + 13 \cdot (q-1,3) + 50585/2] \cdot q^{13} \\
& + [-4085 \cdot (q,2) + 11/2 \cdot (q,3) + 49 \cdot (q-1,3) + 61753/2] \cdot q^{12} \\
& + [-5976 \cdot (q,2) + 11/2 \cdot (q,3) + 134 \cdot (q-1,3) + 4 \cdot (q-1,4) + 75551/2] \cdot q^{11} \\
& + [-8335 \cdot (q,2) + 23/2 \cdot (q,3) + 329 \cdot (q-1,3) + 25 \cdot (q-1,4) + 91017/2] \cdot q^{10} \\
& + [-11296 \cdot (q,2) + 11/2 \cdot (q,3) + 687 \cdot (q-1,3) + 103 \cdot (q-1,4) + 108929/2] \cdot q^{9} \\
& + [-14659 \cdot (q,2) + 11 \cdot (q,3) + 1305 \cdot (q-1,3) + 309 \cdot (q-1,4) + 1/2 \cdot (q,5) + 3 \cdot (q-1,5) \\
& \quad +127313/2] \cdot q^{8} \\
& + [-(q,2) \cdot (q-1,3) - 18388 \cdot (q,2) - 7/2 \cdot (q,3) + 2211 \cdot (q-1,3) + 753 \cdot (q-1,4) \\
& \quad +19 \cdot (q-1,5) - 1/2 \cdot (q-2,5) - 1/2 \cdot (q-3,5) + 145951/2] \cdot q^{7} \\
& + [-24 \cdot (q,2) \cdot (q-1,3) - 21863 \cdot (q,2) + 9/2 \cdot (q,3) + 3480 \cdot (q-1,3) + 1533 \cdot (q-1,4) \\
& \quad +1/2 \cdot (q,5) + 82 \cdot (q-1,5) + 1/3 \cdot (q-1,7) + 1/3 \cdot (q-2,7) + 1/3 \cdot (q-4,7) + 80374] \cdot q^{6} \\
& + [-145 \cdot (q,2) \cdot (q-1,3) - 24477 \cdot (q,2) - 15 \cdot (q,3) + 5117 \cdot (q-1,3) + 2680 \cdot (q-1,4) \\
& \quad -5/4 \cdot (q,5) + 237 \cdot (q-1,5) - (q-2,5) - (q-3,5) + 6 \cdot (q-1,7) + 4 \cdot (q-1,8) + 337961/4] \cdot q^{5} \\
& + [-492 \cdot (q,2) \cdot (q-1,3) - 25063 \cdot (q,2) - 9 \cdot (q,3) + 7102 \cdot (q-1,3) + 3991 \cdot (q-1,4) \\
& \quad +5/4 \cdot (q,5) + 507 \cdot (q-1,5) + (q-2,5) + (q-3,5) + 33 \cdot (q-1,7) + 32 \cdot (q-1,8) + 329851/4] \cdot q^{4} \\
& + [-1022 \cdot (q,2) \cdot (q-1,3) - 22705 \cdot (q,2) - 69/2 \cdot (q,3) + 8611 \cdot (q-1,3) + 4886 \cdot (q-1,4) \\
& \quad -5/4 \cdot (q,5) + 762 \cdot (q-1,5) - (q-2,5) - (q-3,5) + 84 \cdot (q-1,7) + (q-2,7) + (q-4,7) \\
& \quad +104 \cdot (q-1,8) + 290639/4] \cdot q^{3} \\
& + [-1279 \cdot (q,2) \cdot (q-1,3) - 16998 \cdot (q,2) - 39 \cdot (q,3) + 7961 \cdot (q-1,3) + 4496 \cdot (q-1,4) \\
& \quad +7/4 \cdot (q,5) + 753 \cdot (q-1,5) + 3/2 \cdot (q-2,5) + 3/2 \cdot (q-3,5) + 109 \cdot (q-1,7) + 168 \cdot (q-1,8) \\
& \quad +215485/4] \cdot q^{2} \\
& + [-870 \cdot (q,2) \cdot (q-1,3) - 9256 \cdot (q,2) - 32 \cdot (q,3) + 4590 \cdot (q-1,3) + 2649 \cdot (q-1,4) \\
& \quad -1/2 \cdot (q,5) + 418 \cdot (q-1,5) - 1/2 \cdot (q-2,5) - 1/2 \cdot (q-3,5) + 70 \cdot (q-1,7) + 132 \cdot (q-1,8) \\
& \quad +58735/2] \cdot q \\
& + [-240 \cdot (q,2) \cdot (q-1,3) - 2788 \cdot (q,2) - 35/2 \cdot (q,3) + 1169 \cdot (q-1,3) + 721 \cdot (q-1,4) \\
& \quad +3/4 \cdot (q,5) + 99 \cdot (q-1,5) + 1/2 \cdot (q-2,5) + 1/2 \cdot (q-3,5) + 53/3 \cdot (q-1,7) + 2/3 \cdot (q-2,7) \\
& \quad +2/3 \cdot (q-4,7) + 40 \cdot (q-1,8) + 35235/4]
\end{aligned}
$$

A.4.7 Dimension $d = 11$

$$
\begin{aligned}
N_{11,4}(q) = {} & q^{48} + q^{47} + 3 \cdot q^{46} + 5 \cdot q^{45} + 10 \cdot q^{44} + 15 \cdot q^{43} + 27 \cdot q^{42} + 40 \cdot q^{41} + 64 \cdot q^{40} \\
& + 93 \cdot q^{39} + 139 \cdot q^{38} + 195 \cdot q^{37} + 280 \cdot q^{36} + 381 \cdot q^{35} + 525 \cdot q^{34} \\
& + 699 \cdot q^{33} + 933 \cdot q^{32} + 1212 \cdot q^{31} + 1578 \cdot q^{30} + 2007 \cdot q^{29} + 2552 \cdot q^{28} \\
& + 3188 \cdot q^{27} + 3971 \cdot q^{26} + 4874 \cdot q^{25} + [-(q,2) + 5972] \cdot q^{24} \\
& + [-2 \cdot (q,2) + 7220] \cdot q^{23} + [-11 \cdot (q,2) + 8724] \cdot q^{22} + [-32 \cdot (q,2) + 10451] \cdot q^{21} \\
& + [-91 \cdot (q,2) + 12547] \cdot q^{20} + [-201 \cdot (q,2) + 14990] \cdot q^{19} + [-429 \cdot (q,2) + 18032] \cdot q^{18} \\
& + [-800 \cdot (q,2) + 21648] \cdot q^{17} + [-1438 \cdot (q,2) + 26214] \cdot q^{16}
\end{aligned}
$$

$$
\begin{aligned}
&+ [-2377 \cdot (q,2) + (q,3) + 3 \cdot (q-1,3) + 31703] \cdot q^{15}\\
&+ [-3798 \cdot (q,2) + 3/2 \cdot (q,3) + 13 \cdot (q-1,3) + 77153/2] \cdot q^{14}\\
&+ [-5722 \cdot (q,2) + 11/2 \cdot (q,3) + 50 \cdot (q-1,3) + 93463/2] \cdot q^{13}\\
&+ [-8368 \cdot (q,2) + 7 \cdot (q,3) + 141 \cdot (q-1,3) + 2 \cdot (q-1,4) + 56653] \cdot q^{12}\\
&+ [-11688 \cdot (q,2) + 13 \cdot (q,3) + 354 \cdot (q-1,3) + 18 \cdot (q-1,4) + 67971] \cdot q^{11}\\
&+ [-15881 \cdot (q,2) + 21/2 \cdot (q,3) + 765 \cdot (q-1,3) + 83 \cdot (q-1,4) + 162009/2] \cdot q^{10}\\
&+ [-20741 \cdot (q,2) + 13 \cdot (q,3) + 1499 \cdot (q-1,3) + 277 \cdot (q-1,4) + 1/4 \cdot (q,5) + (q-1,5) + 379539/4] \cdot q^{9}\\
&+ [-26304 \cdot (q,2) + 3 \cdot (q,3) + 2638 \cdot (q-1,3) + 725 \cdot (q-1,4) + 9 \cdot (q-1,5) - 1/2 \cdot (q-2,5)\\
&\quad -1/2 \cdot (q-3,5) + 109351] \cdot q^{8}\\
&+ [-8 \cdot (q,2) \cdot (q-1,3) - 31915 \cdot (q,2) + 3 \cdot (q,3) + 4273 \cdot (q-1,3) + 1597 \cdot (q-1,4) + 3/4 \cdot (q,5)\\
&\quad +52 \cdot (q-1,5) + 489525/4] \cdot q^{7}\\
&+ [-69 \cdot (q,2) \cdot (q-1,3) - 36997 \cdot (q,2) - 21/2 \cdot (q,3) + 6414 \cdot (q-1,3) + 3003 \cdot (q-1,4)\\
&\quad -5/4 \cdot (q,5) + 184 \cdot (q-1,5) - 3/2 \cdot (q-2,5) - 3/2 \cdot (q-3,5) + 4/3 \cdot (q-1,7) + 1/3 \cdot (q-2,7)\\
&\quad +1/3 \cdot (q-4,7) + (q-1,8) + 528895/4] \cdot q^{6}\\
&+ [-331 \cdot (q,2) \cdot (q-1,3) - 40045 \cdot (q,2) - 10 \cdot (q,3) + 9157 \cdot (q-1,3) + 4923 \cdot (q-1,4)\\
&\quad +484 \cdot (q-1,5) + 15 \cdot (q-1,7) + 14 \cdot (q-1,8) + 135141] \cdot q^{5}\\
&+ [-950 \cdot (q,2) \cdot (q-1,3) - 39751 \cdot (q,2) - 63/2 \cdot (q,3) + 12198 \cdot (q-1,3) + 6933 \cdot (q-1,4)\\
&\quad +941 \cdot (q-1,5) + 67 \cdot (q-1,7) + 73 \cdot (q-1,8) + 256887/2] \cdot q^{4}\\
&+ [-1782 \cdot (q,2) \cdot (q-1,3) - 34772 \cdot (q,2) - 37 \cdot (q,3) + 14181 \cdot (q-1,3) + 8048 \cdot (q-1,4)\\
&\quad +1322 \cdot (q-1,5) + 152 \cdot (q-1,7) + (q-2,7) + (q-4,7) + 196 \cdot (q-1,8) + 109554] \cdot q^{3}\\
&+ [-2063 \cdot (q,2) \cdot (q-1,3) - 25113 \cdot (q,2) - 115/2 \cdot (q,3) + 12492 \cdot (q-1,3) + 7043 \cdot (q-1,4)\\
&\quad +3/2 \cdot (q,5) + 1234 \cdot (q-1,5) + 3/2 \cdot (q-2,5) + 3/2 \cdot (q-3,5) + 184 \cdot (q-1,7) + 285 \cdot (q-1,8)\\
&\quad +78597] \cdot q^{2}\\
&+ [-1335 \cdot (q,2) \cdot (q-1,3) - 13171 \cdot (q,2) - 69/2 \cdot (q,3) + 6929 \cdot (q-1,3) + 3966 \cdot (q-1,4)\\
&\quad +661 \cdot (q-1,5) + 113 \cdot (q-1,7) + 210 \cdot (q-1,8) + 82653/2] \cdot q\\
&+ [-354 \cdot (q,2) \cdot (q-1,3) - 3803 \cdot (q,2) - 49/2 \cdot (q,3) + 1705 \cdot (q-1,3) + 1040 \cdot (q-1,4)\\
&\quad +3/4 \cdot (q,5) + 152 \cdot (q-1,5) + 1/2 \cdot (q-2,5) + 1/2 \cdot (q-3,5) + 83/3 \cdot (q-1,7) + 2/3 \cdot (q-2,7)\\
&\quad +2/3 \cdot (q-4,7) + 61 \cdot (q-1,8) + 47731/4]
\end{aligned}
$$

A.4.8 Dimension $d = 12$

$$
\begin{aligned}
N_{12,4}(q) = {}& q^{49} + q^{48} + 3 \cdot q^{47} + 5 \cdot q^{46} + 10 \cdot q^{45} + 15 \cdot q^{44} + 27 \cdot q^{43} + 40 \cdot q^{42} + 65 \cdot q^{41}\\
&+ 93 \cdot q^{40} + 141 \cdot q^{39} + 197 \cdot q^{38} + 285 \cdot q^{37} + 386 \cdot q^{36} + 537 \cdot q^{35}\\
&+ 712 \cdot q^{34} + 958 \cdot q^{33} + 1241 \cdot q^{32} + 1626 \cdot q^{31} + 2065 \cdot q^{30} + 2642 \cdot q^{29}\\
&+ 3294 \cdot q^{28} + 4127 \cdot q^{27} + 5061 \cdot q^{26} + 6228 \cdot q^{25} + [-(q,2) + 7522] \cdot q^{24}\\
&+ [-5 \cdot (q,2) + 9119] \cdot q^{23} + [-18 \cdot (q,2) + 10894] \cdot q^{22} + [-55 \cdot (q,2) + 13086] \cdot q^{21}\\
&+ [-134 \cdot (q,2) + 15558] \cdot q^{20} + [-302 \cdot (q,2) + 18662] \cdot q^{19}\\
&+ [-596 \cdot (q,2) + 22257] \cdot q^{18} + [-1111 \cdot (q,2) + 26823] \cdot q^{17}\\
&+ [-1908 \cdot (q,2) + 1/2 \cdot (q,3) + (q-1,3) + 64439/2] \cdot q^{16}\\
&+ [-3142 \cdot (q,2) + (q,3) + 5 \cdot (q-1,3) + 39074] \cdot q^{15}\\
&+ [-4874 \cdot (q,2) + 4 \cdot (q,3) + 23 \cdot (q-1,3) + 47159] \cdot q^{14}\\
&+ [-7310 \cdot (q,2) + 11/2 \cdot (q,3) + 72 \cdot (q-1,3) + 114391/2] \cdot q^{13}\\
&+ [-10468 \cdot (q,2) + 13 \cdot (q,3) + 202 \cdot (q-1,3) + 3 \cdot (q-1,4) + 68742] \cdot q^{12}\\
&+ [-14559 \cdot (q,2) + 11 \cdot (q,3) + 473 \cdot (q-1,3) + 27 \cdot (q-1,4) + 82444] \cdot q^{11}\\
&+ [-19468 \cdot (q,2) + 19 \cdot (q,3) + 1009 \cdot (q-1,3) + 117 \cdot (q-1,4) + 97389] \cdot q^{10}\\
&+ [-25320 \cdot (q,2) + 6 \cdot (q,3) + 1909 \cdot (q-1,3) + 374 \cdot (q-1,4) - 1/4 \cdot (q,5) + (q-1,5)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -1/2 \cdot (q-2,5) - 1/2 \cdot (q-3,5) + 455485/4] \cdot q^9 \\
&+ [-(q,2) \cdot (q-1,3) - 31678 \cdot (q,2) + 14 \cdot (q,3) + 3322 \cdot (q-1,3) + 947 \cdot (q-1,4) + 3/4 \cdot (q,5) \\
&\quad +14 \cdot (q-1,5) + 520077/4] \cdot q^8 \\
&+ [-13 \cdot (q,2) \cdot (q-1,3) - 38243 \cdot (q,2) - 19/2 \cdot (q,3) + 5263 \cdot (q-1,3) + 2023 \cdot (q-1,4) \\
&\quad -3/4 \cdot (q,5) + 69 \cdot (q-1,5) - 3/2 \cdot (q-2,5) - 3/2 \cdot (q-3,5) + 580081/4] \cdot q^7 \\
&+ [-99 \cdot (q,2) \cdot (q-1,3) - 43777 \cdot (q,2) + 4 \cdot (q,3) + 7841 \cdot (q-1,3) + 3723 \cdot (q-1,4) \\
&\quad +3/4 \cdot (q,5) + 241 \cdot (q-1,5) + 1/2 \cdot (q-2,5) + 1/2 \cdot (q-3,5) \\
&\quad +2 \cdot (q-1,7) + 2 \cdot (q-1,8) + 620469/4] \cdot q^6 \\
&+ [-425 \cdot (q,2) \cdot (q-1,3) - 47039 \cdot (q,2) - 55/2 \cdot (q,3) + 11026 \cdot (q-1,3) + 5982 \cdot (q-1,4) \\
&\quad -2 \cdot (q,5) + 605 \cdot (q-1,5) - 3/2 \cdot (q-2,5) - 3/2 \cdot (q-3,5) + 20 \cdot (q-1,7) + 20 \cdot (q-1,8) \\
&\quad +315233/2] \cdot q^5 \\
&+ [-1176 \cdot (q,2) \cdot (q-1,3) - 46149 \cdot (q,2) - 51/2 \cdot (q,3) + 14552 \cdot (q-1,3) + 8282 \cdot (q-1,4) \\
&\quad +2 \cdot (q,5) + 1151 \cdot (q-1,5) + 3/2 \cdot (q-2,5) + 3/2 \cdot (q-3,5) + 84 \cdot (q-1,7) + 94 \cdot (q-1,8) \\
&\quad +296445/2] \cdot q^4 \\
&+ [-2126 \cdot (q,2) \cdot (q-1,3) - 39952 \cdot (q,2) - 111/2 \cdot (q,3) + 16646 \cdot (q-1,3) + 9460 \cdot (q-1,4) \\
&\quad -3/2 \cdot (q,5) + 1577 \cdot (q-1,5) - 3/2 \cdot (q-2,5) - 3/2 \cdot (q-3,5) + 184 \cdot (q-1,7) + 240 \cdot (q-1,8) \\
&\quad +125299] \cdot q^3 \\
&+ [-2412 \cdot (q,2) \cdot (q-1,3) - 28527 \cdot (q,2) - 55 \cdot (q,3) + 14476 \cdot (q-1,3) + 8146 \cdot (q-1,4) \\
&\quad +5/2 \cdot (q,5) + 1449 \cdot (q-1,5) + 5/2 \cdot (q-2,5) + 5/2 \cdot (q-3,5) + 218 \cdot (q-1,7) + 338 \cdot (q-1,8) \\
&\quad +177771/2] \cdot q^2 \\
&+ [-1532 \cdot (q,2) \cdot (q-1,3) - 14779 \cdot (q,2) - 42 \cdot (q,3) + 7914 \cdot (q-1,3) + 4524 \cdot (q-1,4) \\
&\quad -1/2 \cdot (q,5) + 766 \cdot (q-1,5) + 132 \cdot (q-1,7) + 244 \cdot (q-1,8) + 92469/2] \cdot q \\
&+ [-404 \cdot (q,2) \cdot (q-1,3) - 4207 \cdot (q,2) - 23 \cdot (q,3) + 1936 \cdot (q-1,3) + 1172 \cdot (q-1,4) + (q,5) \\
&\quad +175 \cdot (q-1,5) + 1/2 \cdot (q-2,5) + 1/2 \cdot (q-3,5) + 32 \cdot (q-1,7) + 70 \cdot (q-1,8) + 13164]
\end{aligned}
$$

A.5 Rank five

A.5.1 Dimension $d = 6$

$$N_{6,5}(q) = 2 \cdot q^2 + [-2 \cdot (q,2) + 14] \cdot q + [-12 \cdot (q,2) + 47]$$

A.5.2 Dimension $d = 7$

$$
\begin{aligned}
N_{7,5}(q) = {} & q^{22} + q^{21} + 3 \cdot q^{20} + 4 \cdot q^{19} + 8 \cdot q^{18} + 11 \cdot q^{17} + 18 \cdot q^{16} + 24 \cdot q^{15} \\
&+ [-(q,2) + 39] \cdot q^{14} + [-2 \cdot (q,2) + 54] \cdot q^{13} + [-4 \cdot (q,2) + 79] \cdot q^{12} \\
&+ [-7 \cdot (q,2) + 109] \cdot q^{11} + [-14 \cdot (q,2) + 159] \cdot q^{10} + [-24 \cdot (q,2) + 221] \cdot q^9 \\
&+ [-45 \cdot (q,2) + 319] \cdot q^8 + [-78 \cdot (q,2) + 1/2 \cdot (q,3) + (q-1,3) + 915/2] \cdot q^7 \\
&+ [-131 \cdot (q,2) + 1/2 \cdot (q,3) + 4 \cdot (q-1,3) + (q-1,4) + 1293/2] \cdot q^6 \\
&+ [-207 \cdot (q,2) + (q,3) + 13 \cdot (q-1,3) + 5 \cdot (q-1,4) + 895] \cdot q^5 \\
&+ [-2 \cdot (q,2) \cdot (q-1,3) - 305 \cdot (q,2) + 3/2 \cdot (q,3) + 33 \cdot (q-1,3) + 15 \cdot (q-1,4) - 1/4 \cdot (q,5) \\
&\quad -1/2 \cdot (q-2,5) - 1/2 \cdot (q-3,5) + 4811/4] \cdot q^4
\end{aligned}
$$

$$
\begin{aligned}
&+[-8\cdot(q,2)\cdot(q-1,3)-1697/4\cdot(q,2)+5/2\cdot(q,3)+71\cdot(q-1,3)+67/2\cdot(q-1,4)\\
&\quad+2\cdot(q-1,5)+5/8\cdot(q-1,8)+5/8\cdot(q-5,8)+6149/4]\cdot q^3\\
&+[-19\cdot(q,2)\cdot(q-1,3)-2021/4\cdot(q,2)+110\cdot(q-1,3)+103/2\cdot(q-1,4)-(q,5)\\
&\quad+2\cdot(q-1,5)-(q-2,5)-(q-3,5)+13/8\cdot(q-1,8)+5/8\cdot(q-5,8)+7035/4]\cdot q^2\\
&+[-25\cdot(q,2)\cdot(q-1,3)-955/2\cdot(q,2)-1/2\cdot(q,3)+115\cdot(q-1,3)+51\cdot(q-1,4)\\
&\quad+2\cdot(q-1,5)+9/4\cdot(q-1,8)+5/4\cdot(q-5,8)+1579]\cdot q\\
&+[-11\cdot(q,2)\cdot(q-1,3)-537/2\cdot(q,2)-3\cdot(q,3)+45\cdot(q-1,3)+22\cdot(q-1,4)-1/2\cdot(q,5)\\
&\quad-1/2\cdot(q-2,5)-1/2\cdot(q-3,5)+5/4\cdot(q-1,8)+5/4\cdot(q-5,8)+863]
\end{aligned}
$$

A.5.3 Dimension $d = 8$

$$
\begin{aligned}
N_{8,5}(q) = {}& q^{42}+q^{41}+3\cdot q^{40}+5\cdot q^{39}+9\cdot q^{38}+14\cdot q^{37}+23\cdot q^{36}+33\cdot q^{35}+50\cdot q^{34}\\
&+70\cdot q^{33}+99\cdot q^{32}+134\cdot q^{31}+183\cdot q^{30}+240\cdot q^{29}+317\cdot q^{28}+408\cdot q^{27}\\
&+[-(q,2)+527]\cdot q^{26}+[-2\cdot(q,2)+667]\cdot q^{25}+[-5\cdot(q,2)+849]\cdot q^{24}\\
&+[-9\cdot(q,2)+1060]\cdot q^{23}+[-17\cdot(q,2)+1331]\cdot q^{22}+[-27\cdot(q,2)+1648]\cdot q^{21}\\
&+[-46\cdot(q,2)+2051]\cdot q^{20}+[-73\cdot(q,2)+2526]\cdot q^{19}+[-121\cdot(q,2)+3137]\cdot q^{18}\\
&+[-194\cdot(q,2)+3869]\cdot q^{17}+[-313\cdot(q,2)+4811]\cdot q^{16}+[-488\cdot(q,2)+5957]\cdot q^{15}\\
&+[-753\cdot(q,2)+1/2\cdot(q,3)+(q-1,3)+14849/2]\cdot q^{14}\\
&+[-1127\cdot(q,2)+2\cdot(q,3)+6\cdot(q-1,3)+9211]\cdot q^{13}\\
&+[-1664\cdot(q,2)+3\cdot(q,3)+20\cdot(q-1,3)+2\cdot(q-1,4)+11489]\cdot q^{12}\\
&+[-2384\cdot(q,2)+11/2\cdot(q,3)+56\cdot(q-1,3)+8\cdot(q-1,4)+28421/2]\cdot q^{11}\\
&+[-3355\cdot(q,2)+5\cdot(q,3)+128\cdot(q-1,3)+25\cdot(q-1,4)+17578]\cdot q^{10}\\
&+[-4581\cdot(q,2)+7\cdot(q,3)+262\cdot(q-1,3)+66\cdot(q-1,4)+21445]\cdot q^{9}\\
&+[-2\cdot(q,2)\cdot(q-1,3)-6112\cdot(q,2)+11/2\cdot(q,3)+487\cdot(q-1,3)+157\cdot(q-1,4)\\
&\quad-1/4\cdot(q,5)+(q-1,5)-1/2\cdot(q-2,5)-1/2\cdot(q-3,5)+103715/4]\cdot q^{8}\\
&+[-15\cdot(q,2)\cdot(q-1,3)-7852\cdot(q,2)+21/2\cdot(q,3)+862\cdot(q-1,3)+337\cdot(q-1,4)\\
&\quad-1/4\cdot(q,5)+10\cdot(q-1,5)-1/2\cdot(q-2,5)-1/2\cdot(q-3,5)+122363/4]\cdot q^{7}\\
&+[-55\cdot(q,2)\cdot(q-1,3)-9732\cdot(q,2)+6\cdot(q,3)+1432\cdot(q-1,3)+658\cdot(q-1,4)-5/4\cdot(q,5)\\
&\quad+36\cdot(q-1,5)-3/2\cdot(q-2,5)-3/2\cdot(q-3,5)+1/3\cdot(q-1,7)+1/3\cdot(q-2,7)+1/3\cdot(q-4,7)\\
&\quad+(q-1,8)+140889/4]\cdot q^{6}\\
&+[-167\cdot(q,2)\cdot(q-1,3)-45481/4\cdot(q,2)+11\cdot(q,3)+2281\cdot(q-1,3)+2303/2\cdot(q-1,4)\\
&\quad-5/4\cdot(q,5)+98\cdot(q-1,5)-3/2\cdot(q-2,5)-3/2\cdot(q-3,5)+4\cdot(q-1,7)+57/8\cdot(q-1,8)\\
&\quad+1/2\cdot(q-3,8)+5/8\cdot(q-5,8)+38708]\cdot q^{5}\\
&+[-406\cdot(q,2)\cdot(q-1,3)-4\cdot(q,2)\cdot(q-1,5)-12419\cdot(q,2)+2\cdot(q,3)+3391\cdot(q-1,3)\\
&\quad+1773\cdot(q-1,4)-7/4\cdot(q,5)+211\cdot(q-1,5)-2\cdot(q-2,5)-2\cdot(q-3,5)+55/3\cdot(q-1,7)\\
&\quad+1/3\cdot(q-2,7)+1/3\cdot(q-4,7)+65/2\cdot(q-1,8)+(q-3,8)+5/2\cdot(q-5,8)+2\cdot(q-1,9)\\
&\quad+161255/4]\cdot q^{4}\\
&+[-754\cdot(q,2)\cdot(q-1,3)-20\cdot(q,2)\cdot(q-1,5)-(q,2)\cdot(q-1,7)+1/2\cdot(q,3)\cdot(q-1,4)\\
&\quad+4\cdot(q-1,3)\cdot(q-1,4)-48671/4\cdot(q,2)+5\cdot(q,3)+4438\cdot(q-1,3)+2290\cdot(q-1,4)\\
&\quad-3/2\cdot(q,5)+348\cdot(q-1,5)-3/2\cdot(q-2,5)-3/2\cdot(q-3,5)+131/3\cdot(q-1,7)\\
&\quad+2/3\cdot(q-2,7)+2/3\cdot(q-4,7)+679/8\cdot(q-1,8)+3/2\cdot(q-3,8)+35/8\cdot(q-5,8)+8\cdot(q-1,9)\\
&\quad+153023/4]\cdot q^{3}\\
&+[-961\cdot(q,2)\cdot(q-1,3)-36\cdot(q,2)\cdot(q-1,5)-3\cdot(q,2)\cdot(q-1,7)+1/2\cdot(q,3)\cdot(q-1,4)\\
&\quad+12\cdot(q-1,3)\cdot(q-1,4)-20293/2\cdot(q,2)-31/2\cdot(q,3)+4476\cdot(q-1,3)\\
&\quad+4511/2\cdot(q-1,4)-3/2\cdot(q,5)+382\cdot(q-1,5)-3/2\cdot(q-2,5)-3/2\cdot(q-3,5)\\
&\quad+172/3\cdot(q-1,7)+1/3\cdot(q-2,7)+1/3\cdot(q-4,7)+517/4\cdot(q-1,8)+5/2\cdot(q-3,8)\\
&\quad+31/4\cdot(q-5,8)+12\cdot(q-1,9)+62691/2]\cdot q^{2}
\end{aligned}
$$

$$
\begin{aligned}
&+[-716 \cdot (q,2) \cdot (q-1,3) - 28 \cdot (q,2) \cdot (q-1,5) - 3 \cdot (q,2) \cdot (q-1,7) \\
&\quad +12 \cdot (q-1,3) \cdot (q-1,4) - 25373/4 \cdot (q,2) - 37/2 \cdot (q,3) + 2897 \cdot (q-1,3) \\
&\quad +2923/2 \cdot (q-1,4) - 1/2 \cdot (q,5) + 236 \cdot (q-1,5) - 1/2 \cdot (q-2,5) - 1/2 \cdot (q-3,5) \\
&\quad +39 \cdot (q-1,7) + 845/8 \cdot (q-1,8) + (q-3,8) + 69/8 \cdot (q-5,8) + 8 \cdot (q-1,9) + 77079/4] \cdot q \\
&+[-221 \cdot (q,2) \cdot (q-1,3) - 8 \cdot (q,2) \cdot (q-1,5) - (q,2) \cdot (q-1,7) + 4 \cdot (q-1,3) \cdot (q-1,4) \\
&\quad -4451/2 \cdot (q,2) - 35/2 \cdot (q,3) + 827 \cdot (q-1,3) + 423 \cdot (q-1,4) - 1/2 \cdot (q,5) + 60 \cdot (q-1,5) \\
&\quad -1/2 \cdot (q-2,5) - 1/2 \cdot (q-3,5) + 32/3 \cdot (q-1,7) + 2/3 \cdot (q-2,7) + 2/3 \cdot (q-4,7) \\
&\quad +157/4 \cdot (q-1,8) + 3/2 \cdot (q-3,8) + 39/4 \cdot (q-5,8) + 2 \cdot (q-1,9) + 13345/2]
\end{aligned}
$$

A.5.4 Dimension $d = 9$

$$
\begin{aligned}
N_{9,5}(q) = {} & q^{60} + q^{59} + 3 \cdot q^{58} + 5 \cdot q^{57} + 10 \cdot q^{56} + 15 \cdot q^{55} + 26 \cdot q^{54} + 38 \cdot q^{53} + 60 \cdot q^{52} \\
&+ 85 \cdot q^{51} + 125 \cdot q^{50} + 172 \cdot q^{49} + 243 \cdot q^{48} + 325 \cdot q^{47} + 442 \cdot q^{46} \\
&+ 580 \cdot q^{45} + 767 \cdot q^{44} + 986 \cdot q^{43} + 1275 \cdot q^{42} + 1612 \cdot q^{41} + 2045 \cdot q^{40} \\
&+ 2548 \cdot q^{39} + 3178 \cdot q^{38} + 3908 \cdot q^{37} + [-(q,2) + 4810] \cdot q^{36} \\
&+ [-2 \cdot (q,2) + 5845] \cdot q^{35} + [-5 \cdot (q,2) + 7105] \cdot q^{34} + [-10 \cdot (q,2) + 8553] \cdot q^{33} \\
&+ [-20 \cdot (q,2) + 10291] \cdot q^{32} + [-35 \cdot (q,2) + 12282] \cdot q^{31} + [-62 \cdot (q,2) + 14657] \cdot q^{30} \\
&+ [-103 \cdot (q,2) + 17374] \cdot q^{29} + [-170 \cdot (q,2) + 20595] \cdot q^{28} \\
&+ [-273 \cdot (q,2) + 24294] \cdot q^{27} + [-435 \cdot (q,2) + 28666] \cdot q^{26} \\
&+ [-679 \cdot (q,2) + 33713] \cdot q^{25} + [-1054 \cdot (q,2) + 39702] \cdot q^{24} \\
&+ [-1600 \cdot (q,2) + 46650] \cdot q^{23} + [-2405 \cdot (q,2) + 54925] \cdot q^{22} \\
&+ [-3544 \cdot (q,2) + 64612] \cdot q^{21} + [-5156 \cdot (q,2) + 1/2 \cdot (q,3) + (q-1,3) + 152331/2] \cdot q^{20} \\
&+ [-7362 \cdot (q,2) + 3 \cdot (q,3) + 8 \cdot (q-1,3) + 89755] \cdot q^{19} \\
&+ [-10377 \cdot (q,2) + 11/2 \cdot (q,3) + 28 \cdot (q-1,3) + 211959/2] \cdot q^{18} \\
&+ [-14372 \cdot (q,2) + 12 \cdot (q,3) + 86 \cdot (q-1,3) + 3 \cdot (q-1,4) + 125045] \cdot q^{17} \\
&+ [-19643 \cdot (q,2) + 29/2 \cdot (q,3) + 211 \cdot (q-1,3) + 14 \cdot (q-1,4) + 295333/2] \cdot q^{16} \\
&+ [-26415 \cdot (q,2) + 24 \cdot (q,3) + 477 \cdot (q-1,3) + 47 \cdot (q-1,4) + 174065] \cdot q^{15} \\
&+ [-35014 \cdot (q,2) + 49/2 \cdot (q,3) + 961 \cdot (q-1,3) + 128 \cdot (q-1,4) + 409677/2] \cdot q^{14} \\
&+ [-45656 \cdot (q,2) + 73/2 \cdot (q,3) + 1818 \cdot (q-1,3) + 312 \cdot (q-1,4) + 480013/2] \cdot q^{13} \\
&+ [-58589 \cdot (q,2) + 61/2 \cdot (q,3) + 3193 \cdot (q-1,3) + 691 \cdot (q-1,4) + 1/4 \cdot (q,5) + (q-1,5) \\
&\quad +1118993/4] \cdot q^{12} \\
&+ [-5 \cdot (q,2) \cdot (q-1,3) - 73762 \cdot (q,2) + 87/2 \cdot (q,3) + 5347 \cdot (q-1,3) + 1427 \cdot (q-1,4) \\
&\quad +1/2 \cdot (q,5) + 10 \cdot (q-1,5) + 323106] \cdot q^{11} \\
&+ [-35 \cdot (q,2) \cdot (q-1,3) - 91006 \cdot (q,2) + 65/2 \cdot (q,3) + 8518 \cdot (q-1,3) + 2756 \cdot (q-1,4) \\
&\quad -3/4 \cdot (q,5) + 46 \cdot (q-1,5) - 3/2 \cdot (q-2,5) - 3/2 \cdot (q-3,5) + 1475957/4] \cdot q^{10} \\
&+ [-148 \cdot (q,2) \cdot (q-1,3) - 109596 \cdot (q,2) + 93/2 \cdot (q,3) + 13105 \cdot (q-1,3) + 5019 \cdot (q-1,4) \\
&\quad +1/2 \cdot (q,5) + 167 \cdot (q-1,5) - 1/2 \cdot (q-2,5) - 1/2 \cdot (q-3,5) + 1/3 \cdot (q-1,7) + 1/3 \cdot (q-2,7) \\
&\quad +1/3 \cdot (q-4,7) + (q-1,8) + 414478] \cdot q^{9} \\
&+ [-495 \cdot (q,2) \cdot (q-1,3) - 128263 \cdot (q,2) + 29 \cdot (q,3) + 19506 \cdot (q-1,3) + 8602 \cdot (q-1,4) \\
&\quad -4 \cdot (q,5) + 472 \cdot (q-1,5) - 9/2 \cdot (q-2,5) - 9/2 \cdot (q-3,5) + 19/3 \cdot (q-1,7) + 1/3 \cdot (q-2,7) \\
&\quad +1/3 \cdot (q-4,7) + 8 \cdot (q-1,8) + 455715] \cdot q^{8} \\
&+ [-1368 \cdot (q,2) \cdot (q-1,3) - 144829 \cdot (q,2) + 44 \cdot (q,3) + 28273 \cdot (q-1,3) + 13840 \cdot (q-1,4) \\
&\quad -5/4 \cdot (q,5) + 1123 \cdot (q-1,5) - 2 \cdot (q-2,5) - 2 \cdot (q-3,5) + 113/3 \cdot (q-1,7) + 2/3 \cdot (q-2,7) \\
&\quad +2/3 \cdot (q-4,7) + 45 \cdot (q-1,8) + 1945617/4] \cdot q^{7} \\
&+ [-3190 \cdot (q,2) \cdot (q-1,3) - 6 \cdot (q,2) \cdot (q-1,5) - 312235/2 \cdot (q,2) + 11/2 \cdot (q,3) \\
&\quad +39566 \cdot (q-1,3) + 20678 \cdot (q-1,4) - 33/4 \cdot (q,5) + 2260 \cdot (q-1,5) - 19/2 \cdot (q-2,5) \\
&\quad -19/2 \cdot (q-3,5) - 1/2 \cdot (q,7) + 147 \cdot (q-1,7) + 1/2 \cdot (q-2,7) - 1/2 \cdot (q-3,7) + 1/2 \cdot (q-4,7)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -1/2 \cdot (q-5,7) + 739/4 \cdot (q-1,8) + 3/2 \cdot (q-3,8) + 5/4 \cdot (q-5,8) + 6 \cdot (q-1,9) + 1995967/4] \cdot q^6 \\
&+ [-6372 \cdot (q,2) \cdot (q-1,3) - 60 \cdot (q,2) \cdot (q-1,5) + 3 \cdot (q-1,3) \cdot (q-1,4) \\
&\quad -1265453/8 \cdot (q,2) - 11/2 \cdot (q,3) + 52874 \cdot (q-1,3) + 113087/4 \cdot (q-1,4) - 3 \cdot (q,5) \\
&\quad +4011 \cdot (q-1,5) - 3 \cdot (q-2,5) - 3 \cdot (q-3,5) + 421 \cdot (q-1,7) + (q-2,7) + (q-4,7) \\
&\quad +9165/16 \cdot (q-1,8) + (q-3,8) + 109/16 \cdot (q-5,8) + 48 \cdot (q-1,9) + 2 \cdot (q-1,11) + 3876475/8] \cdot q^5 \\
&+ [-10664 \cdot (q,2) \cdot (q-1,3) - 268 \cdot (q,2) \cdot (q-1,5) - 9 \cdot (q,2) \cdot (q-1,7) \\
&\quad +5/2 \cdot (q,3) \cdot (q-1,4) - 1/4 \cdot (q,3) \cdot (q-2,5) - 1/4 \cdot (q,3) \cdot (q-3,5) \\
&\quad +55 \cdot (q-1,3) \cdot (q-1,4) - 1/4 \cdot (q-1,3) \cdot (q,5) - 1/2 \cdot (q-1,3) \cdot (q-2,5) \\
&\quad -1/2 \cdot (q-1,3) \cdot (q-3,5) - 1177433/8 \cdot (q,2) - 59 \cdot (q,3) + 260121/4 \cdot (q-1,3) \\
&\quad +137185/4 \cdot (q-1,4) - 47/4 \cdot (q,5) + 6158 \cdot (q-1,5) - 43/4 \cdot (q-2,5) - 43/4 \cdot (q-3,5) \\
&\quad -2 \cdot (q,7) + 2669/3 \cdot (q-1,7) - 1/3 \cdot (q-2,7) - 2 \cdot (q-3,7) - 1/3 \cdot (q-4,7) - 2 \cdot (q-5,7) \\
&\quad +21401/16 \cdot (q-1,8) + 11/2 \cdot (q-3,8) + 209/16 \cdot (q-5,8) + 503/3 \cdot (q-1,9) - 1/2 \cdot (q-2,9) \\
&\quad -5/6 \cdot (q-4,9) - 1/2 \cdot (q-5,9) - 5/6 \cdot (q-7,9) + 12 \cdot (q-1,11) + (q-1,13) + 3494705/8] \cdot q^4 \\
&+ [-14056 \cdot (q,2) \cdot (q-1,3) - 619 \cdot (q,2) \cdot (q-1,5) - 127/3 \cdot (q,2) \cdot (q-1,7) \\
&\quad -1/3 \cdot (q,2) \cdot (q-2,7) - 1/3 \cdot (q,2) \cdot (q-4,7) + 9/2 \cdot (q,3) \cdot (q-1,4) \\
&\quad +210 \cdot (q-1,3) \cdot (q-1,4) + (q-1,3) \cdot (q-1,5) - 242557/2 \cdot (q,2) - 88 \cdot (q,3) \\
&\quad +68488 \cdot (q-1,3) + 69845/2 \cdot (q-1,4) - 5/2 \cdot (q,5) + 7607 \cdot (q-1,5) - 5/2 \cdot (q-2,5) \\
&\quad -5/2 \cdot (q-3,5) + 3994/3 \cdot (q-1,7) + 10/3 \cdot (q-2,7) + 10/3 \cdot (q-4,7) + 8867/4 \cdot (q-1,8) \\
&\quad +5/2 \cdot (q-3,8) + 105/4 \cdot (q-5,8) + 314 \cdot (q-1,9) + 28 \cdot (q-1,11) + 4 \cdot (q-1,13) + 3 \cdot (q-1,16) \\
&\quad +353232] \cdot q^3 \\
&+ [-13105 \cdot (q,2) \cdot (q-1,3) - 759 \cdot (q,2) \cdot (q-1,5) - 73 \cdot (q,2) \cdot (q-1,7) \\
&\quad +3/2 \cdot (q,3) \cdot (q-1,4) + 331 \cdot (q-1,3) \cdot (q-1,4) - 1/2 \cdot (q-1,3) \cdot (q,5) \\
&\quad +(q-1,3) \cdot (q-1,5) - 1/2 \cdot (q-1,3) \cdot (q-2,5) - 1/2 \cdot (q-1,3) \cdot (q-3,5) \\
&\quad -332009/4 \cdot (q,2) - 160 \cdot (q,3) + 109543/2 \cdot (q-1,3) + 27242 \cdot (q-1,4) - 19/2 \cdot (q,5) \\
&\quad +6562 \cdot (q-1,5) - 19/2 \cdot (q-2,5) - 19/2 \cdot (q-3,5) - 5/2 \cdot (q,7) + 3818/3 \cdot (q-1,7) \\
&\quad -11/6 \cdot (q-2,7) - 5/2 \cdot (q-3,7) - 11/6 \cdot (q-4,7) - 5/2 \cdot (q-5,7) + 19053/8 \cdot (q-1,8) \\
&\quad +7 \cdot (q-3,8) + 269/8 \cdot (q-5,8) + 958/3 \cdot (q-1,9) - (q-2,9) - 5/3 \cdot (q-4,9) - (q-5,9) \\
&\quad -5/3 \cdot (q-7,9) + 32 \cdot (q-1,11) + 6 \cdot (q-1,13) + 9 \cdot (q-1,16) + 961003/4] \cdot q^2 \\
&+ [-7410 \cdot (q,2) \cdot (q-1,3) - 469 \cdot (q,2) \cdot (q-1,5) - 55 \cdot (q,2) \cdot (q-1,7) - (q,3) \cdot (q-1,4) \\
&\quad +236 \cdot (q-1,3) \cdot (q-1,4) + (q-1,3) \cdot (q-1,5) - 332621/8 \cdot (q,2) - 118 \cdot (q,3) \\
&\quad +28043 \cdot (q-1,3) + 55635/4 \cdot (q-1,4) - 3/4 \cdot (q,5) + 3353 \cdot (q-1,5) - 1/2 \cdot (q-2,5) \\
&\quad -1/2 \cdot (q-3,5) + 687 \cdot (q-1,7) + (q-2,7) + (q-4,7) + 23445/16 \cdot (q-1,8) + 3/2 \cdot (q-3,8) \\
&\quad +653/16 \cdot (q-5,8) + 170 \cdot (q-1,9) + 18 \cdot (q-1,11) + 4 \cdot (q-1,13) + 9 \cdot (q-1,16) + 959069/8] \cdot q \\
&+ [-1846 \cdot (q,2) \cdot (q-1,3) - 115 \cdot (q,2) \cdot (q-1,5) - 46/3 \cdot (q,2) \cdot (q-1,7) \\
&\quad -1/3 \cdot (q,2) \cdot (q-2,7) - 1/3 \cdot (q,2) \cdot (q-4,7) - 1/2 \cdot (q,3) \cdot (q-1,4) \\
&\quad -1/4 \cdot (q,3) \cdot (q-2,5) - 1/4 \cdot (q,3) \cdot (q-3,5) + 63 \cdot (q-1,3) \cdot (q-1,4) \\
&\quad -1/4 \cdot (q-1,3) \cdot (q,5) - 1/2 \cdot (q-1,3) \cdot (q-2,5) - 1/2 \cdot (q-1,3) \cdot (q-3,5) \\
&\quad -91433/8 \cdot (q,2) - 75 \cdot (q,3) + 26353/4 \cdot (q-1,3) + 13153/4 \cdot (q-1,4) - 13/4 \cdot (q,5) \\
&\quad +740 \cdot (q-1,5) - 9/4 \cdot (q-2,5) - 9/4 \cdot (q-3,5) - (q,7) + 467/3 \cdot (q-1,7) + 5/3 \cdot (q-2,7) \\
&\quad -(q-3,7) + 5/3 \cdot (q-4,7) - (q-5,7) + 6449/16 \cdot (q-1,8) + 3 \cdot (q-3,8) + 673/16 \cdot (q-5,8) \\
&\quad +107/3 \cdot (q-1,9) - 1/2 \cdot (q-2,9) - 5/6 \cdot (q-4,9) - 1/2 \cdot (q-5,9) - 5/6 \cdot (q-7,9) \\
&\quad +4 \cdot (q-1,11) + (q-1,13) + 3 \cdot (q-1,16) + 263389/8]
\end{aligned}
$$

A.5.5 Dimension $d = 10$

$$
\begin{aligned}
N_{10,5}(q) = {} & q^{76} + q^{75} + 3 \cdot q^{74} + 5 \cdot q^{73} + 10 \cdot q^{72} + 16 \cdot q^{71} + 27 \cdot q^{70} + 41 \cdot q^{69} + 65 \cdot q^{68} \\
& + 95 \cdot q^{67} + 141 \cdot q^{66} + 199 \cdot q^{65} + 284 \cdot q^{64} + 390 \cdot q^{63} + 537 \cdot q^{62} \\
& + 721 \cdot q^{61} + 966 \cdot q^{60} + 1270 \cdot q^{59} + 1665 \cdot q^{58} + 2149 \cdot q^{57} + 2766 \cdot q^{56} \\
& + 3513 \cdot q^{55} + 4447 \cdot q^{54} + 5570 \cdot q^{53} + 6951 \cdot q^{52} + 8595 \cdot q^{51} + 10588 \cdot q^{50}
\end{aligned}
$$

$+\, 12942 \cdot q^{49} + 15761 \cdot q^{48} + 19061 \cdot q^{47} + 22970 \cdot q^{46} + 27509 \cdot q^{45}$

$+\, [-(q,2) + 32837] \cdot q^{44} + [-2 \cdot (q,2) + 38978] \cdot q^{43} + [-6 \cdot (q,2) + 46122] \cdot q^{42}$

$+\, [-13 \cdot (q,2) + 54310] \cdot q^{41} + [-28 \cdot (q,2) + 63767] \cdot q^{40} + [-53 \cdot (q,2) + 74543] \cdot q^{39}$

$+\, [-98 \cdot (q,2) + 86919] \cdot q^{38} + [-169 \cdot (q,2) + 100963] \cdot q^{37}$

$+\, [-289 \cdot (q,2) + 117024] \cdot q^{36} + [-473 \cdot (q,2) + 135193] \cdot q^{35}$

$+\, [-766 \cdot (q,2) + 155920] \cdot q^{34} + [-1209 \cdot (q,2) + 179353] \cdot q^{33}$

$+\, [-1887 \cdot (q,2) + 206067] \cdot q^{32} + [-2883 \cdot (q,2) + 236299] \cdot q^{31}$

$+\, [-4356 \cdot (q,2) + 270817] \cdot q^{30} + [-6464 \cdot (q,2) + 310000] \cdot q^{29}$

$+\, [-9476 \cdot (q,2) + 354882] \cdot q^{28} + [-13670 \cdot (q,2) + 406043] \cdot q^{27}$

$+\, [-19480 \cdot (q,2) + 464878] \cdot q^{26}$

$+\, [-27350 \cdot (q,2) + 3/2 \cdot (q,3) + 3 \cdot (q-1,3) + 1064495/2] \cdot q^{25}$

$+\, [-37936 \cdot (q,2) + 4 \cdot (q,3) + 14 \cdot (q-1,3) + 609989] \cdot q^{24}$

$+\, [-51895 \cdot (q,2) + 11 \cdot (q,3) + 53 \cdot (q-1,3) + 699262] \cdot q^{23}$

$+\, [-70149 \cdot (q,2) + 20 \cdot (q,3) + 152 \cdot (q-1,3) + (q-1,4) + 802396] \cdot q^{22}$

$+\, [-93608 \cdot (q,2) + 73/2 \cdot (q,3) + 387 \cdot (q-1,3) + 8 \cdot (q-1,4) + 1841685/2] \cdot q^{21}$

$+\, [-123489 \cdot (q,2) + 51 \cdot (q,3) + 874 \cdot (q-1,3) + 36 \cdot (q-1,4) + 1057333] \cdot q^{20}$

$+\, [-160938 \cdot (q,2) + 155/2 \cdot (q,3) + 1829 \cdot (q-1,3) + 111 \cdot (q-1,4) + 2426779/2] \cdot q^{19}$

$+\, [-207420 \cdot (q,2) + 94 \cdot (q,3) + 3554 \cdot (q-1,3) + 301 \cdot (q-1,4) + 1391830] \cdot q^{18}$

$+\, [-264157 \cdot (q,2) + 247/2 \cdot (q,3) + 6535 \cdot (q-1,3) + 726 \cdot (q-1,4) + 3187083/2] \cdot q^{17}$

$+\, [-332586 \cdot (q,2) + 133 \cdot (q,3) + 11385 \cdot (q-1,3) + 1612 \cdot (q-1,4) + 1/4 \cdot (q,5) + 7281983/4] \cdot q^{16}$

$+\, [-413478 \cdot (q,2) + 159 \cdot (q,3) + 18969 \cdot (q-1,3) + 3338 \cdot (q-1,4) + 1/2 \cdot (q,5) + 2 \cdot (q-1,5)$
$\quad +4142779/2] \cdot q^{15}$

$+\, [-4 \cdot (q,2) \cdot (q-1,3) - 507500 \cdot (q,2) + 309/2 \cdot (q,3) + 30279 \cdot (q-1,3) + 6518 \cdot (q-1,4)$
$\quad +1/4 \cdot (q,5) + 18 \cdot (q-1,5) - (q-2,5) - (q-3,5) + 9380773/4] \cdot q^{14}$

$+\, [-35 \cdot (q,2) \cdot (q-1,3) - 613896 \cdot (q,2) + 357/2 \cdot (q,3) + 46585 \cdot (q-1,3) + 12081 \cdot (q-1,4)$
$\quad +(q,5) + 93 \cdot (q-1,5) - (q-2,5) - (q-3,5) + 5271237/2] \cdot q^{13}$

$+\, [-176 \cdot (q,2) \cdot (q-1,3) - 731120 \cdot (q,2) + 319/2 \cdot (q,3) + 69267 \cdot (q-1,3) + 21351 \cdot (q-1,4)$
$\quad -5/4 \cdot (q,5) + 348 \cdot (q-1,5) - 7/2 \cdot (q-2,5) - 7/2 \cdot (q-3,5) + 11740051/4] \cdot q^{12}$

$+\, [-689 \cdot (q,2) \cdot (q-1,3) - 854875 \cdot (q,2) + 190 \cdot (q,3) + 100125 \cdot (q-1,3) + 36074 \cdot (q-1,4)$
$\quad -3/4 \cdot (q,5) + 1064 \cdot (q-1,5) - 7/2 \cdot (q-2,5) - 7/2 \cdot (q-3,5) + 4 \cdot (q-1,7) + 3 \cdot (q-1,8)$
$\quad +12908915/4] \cdot q^{11}$

$+\, [-2175 \cdot (q,2) \cdot (q-1,3) - 979050 \cdot (q,2) + 307/2 \cdot (q,3) + 141126 \cdot (q-1,3)$
$\quad +58294 \cdot (q-1,4) - 29/4 \cdot (q,5) + 2733 \cdot (q-1,5) - 10 \cdot (q-2,5) - 10 \cdot (q-3,5) - 1/2 \cdot (q,7)$
$\quad +88/3 \cdot (q-1,7) - 1/6 \cdot (q-2,7) - 1/2 \cdot (q-3,7) - 1/6 \cdot (q-4,7) - 1/2 \cdot (q-5,7)$
$\quad +28 \cdot (q-1,8) + 13972397/4] \cdot q^{10}$

$+\, [-5839 \cdot (q,2) \cdot (q-1,3) - 1093020 \cdot (q,2) + 339/2 \cdot (q,3) + 194733 \cdot (q-1,3)$
$\quad +90003 \cdot (q-1,4) - 13/2 \cdot (q,5) + 6152 \cdot (q-1,5) - 9 \cdot (q-2,5) - 9 \cdot (q-3,5) - 1/2 \cdot (q,7)$
$\quad +430/3 \cdot (q-1,7) - 1/6 \cdot (q-2,7) - 1/2 \cdot (q-3,7) - 1/6 \cdot (q-4,7) - 1/2 \cdot (q-5,7)$
$\quad +141 \cdot (q-1,8) + 7403407/2] \cdot q^{9}$

$+\, [-13658 \cdot (q,2) \cdot (q-1,3) - 2 \cdot (q,2) \cdot (q-1,5) - 1183104 \cdot (q,2) + 79 \cdot (q,3)$
$\quad +263038 \cdot (q-1,3) + 132255 \cdot (q-1,4) - 69/4 \cdot (q,5) + 12261 \cdot (q-1,5) - 19 \cdot (q-2,5)$
$\quad -19 \cdot (q-3,5) - 5/2 \cdot (q,7) + 528 \cdot (q-1,7) - 3/2 \cdot (q-2,7) - 5/2 \cdot (q-3,7) - 3/2 \cdot (q-4,7)$
$\quad -5/2 \cdot (q-5,7) + 554 \cdot (q-1,8) + 1/2 \cdot (q-3,8) + 1/2 \cdot (q-5,8) + 14/3 \cdot (q-1,9)$
$\quad -1/2 \cdot (q-2,9) - 5/6 \cdot (q-4,9) - 1/2 \cdot (q-5,9) - 5/6 \cdot (q-7,9) + 15273747/4] \cdot q^{8}$

$+\, [-28195 \cdot (q,2) \cdot (q-1,3) - 51 \cdot (q,2) \cdot (q-1,5) + 1/2 \cdot (q,3) \cdot (q-1,4)$
$\quad +(q-1,3) \cdot (q-1,4) - 9844109/8 \cdot (q,2) + 49 \cdot (q,3) + 346632 \cdot (q-1,3)$
$\quad +734513/4 \cdot (q-1,4) - 13 \cdot (q,5) + 21964 \cdot (q-1,5) - 33/2 \cdot (q-2,5) - 33/2 \cdot (q-3,5)$
$\quad -5/2 \cdot (q,7) + 1566 \cdot (q-1,7) - 3/2 \cdot (q-2,7) - 5/2 \cdot (q-3,7) - 3/2 \cdot (q-4,7)$
$\quad -5/2 \cdot (q-5,7) + 28325/16 \cdot (q-1,8) + 3/2 \cdot (q-3,8) + 45/16 \cdot (q-5,8) + 197/3 \cdot (q-1,9)$

$$
\begin{aligned}
&-1/2\cdot(q-2,9)-5/6\cdot(q-4,9)-1/2\cdot(q-5,9)-5/6\cdot(q-7,9)+(q-1,11)+30390355/8]\cdot q^7\\
&+[-51502\cdot(q,2)\cdot(q-1,3)-388\cdot(q,2)\cdot(q-1,5)+1/2\cdot(q,2)\cdot(q-2,5)\\
&\quad+1/2\cdot(q,2)\cdot(q-3,5)+(q,3)\cdot(q-1,4)+30\cdot(q-1,3)\cdot(q-1,4)-9723777/8\cdot(q,2)\\
&\quad-131\cdot(q,3)+440769\cdot(q-1,3)+952287/4\cdot(q-1,4)-103/4\cdot(q,5)+35690\cdot(q-1,5)\\
&\quad-57/2\cdot(q-2,5)-57/2\cdot(q-3,5)-5\cdot(q,7)+11423/3\cdot(q-1,7)-10/3\cdot(q-2,7)\\
&\quad-5\cdot(q-3,7)-10/3\cdot(q-4,7)-5\cdot(q-5,7)+75281/16\cdot(q-1,8)+6\cdot(q-3,8)\\
&\quad+161/16\cdot(q-5,8)+375\cdot(q-1,9)-3/2\cdot(q-2,9)-5/2\cdot(q-4,9)-3/2\cdot(q-5,9)\\
&\quad-5/2\cdot(q-7,9)+86/5\cdot(q-1,11)+1/5\cdot(q-3,11)+1/5\cdot(q-4,11)+1/5\cdot(q-5,11)\\
&\quad+1/5\cdot(q-9,11)+28832753/8]\cdot q^6\\
&+[-82455\cdot(q,2)\cdot(q-1,3)-1652\cdot(q,2)\cdot(q-1,5)-36\cdot(q,2)\cdot(q-1,7)\\
&\quad+15/2\cdot(q,3)\cdot(q-1,4)+317\cdot(q-1,3)\cdot(q-1,4)-1/2\cdot(q-1,3)\cdot(q,5)\\
&\quad+(q-1,3)\cdot(q-1,5)-1/2\cdot(q-1,3)\cdot(q-2,5)-1/2\cdot(q-1,3)\cdot(q-3,5)\\
&\quad-8959987/8\cdot(q,2)-214\cdot(q,3)+1058997/2\cdot(q-1,3)+1130399/4\cdot(q-1,4)\\
&\quad-20\cdot(q,5)+52633\cdot(q-1,5)-39/2\cdot(q-2,5)-39/2\cdot(q-3,5)-9/2\cdot(q,7)\\
&\quad+22829/3\cdot(q-1,7)-17/6\cdot(q-2,7)-9/2\cdot(q-3,7)-17/6\cdot(q-4,7)-9/2\cdot(q-5,7)\\
&\quad+164651/16\cdot(q-1,8)+8\cdot(q-3,8)+379/16\cdot(q-5,8)+1277\cdot(q-1,9)-3/2\cdot(q-2,9)\\
&\quad-5/2\cdot(q-4,9)-3/2\cdot(q-5,9)-5/2\cdot(q-7,9)+509/5\cdot(q-1,11)-1/5\cdot(q-3,11)\\
&\quad-1/5\cdot(q-4,11)-1/5\cdot(q-5,11)-1/5\cdot(q-9,11)+6\cdot(q-1,13)+25661301/8]\cdot q^5\\
&+[-111750\cdot(q,2)\cdot(q-1,3)-4333\cdot(q,2)\cdot(q-1,5)-230\cdot(q,2)\cdot(q-1,7)\\
&\quad+29/2\cdot(q,3)\cdot(q-1,4)-1/4\cdot(q,3)\cdot(q-2,5)-1/4\cdot(q,3)\cdot(q-3,5)\\
&\quad+1337\cdot(q-1,3)\cdot(q-1,4)-3/4\cdot(q-1,3)\cdot(q,5)+8\cdot(q-1,3)\cdot(q-1,5)\\
&\quad-(q-1,3)\cdot(q-2,5)-(q-1,3)\cdot(q-3,5)-7513383/8\cdot(q,2)-863/2\cdot(q,3)\\
&\quad+2302123/4\cdot(q-1,3)+1186999/4\cdot(q-1,4)-23\cdot(q,5)+67961\cdot(q-1,5)\\
&\quad-87/4\cdot(q-2,5)-87/4\cdot(q-3,5)-11/2\cdot(q,7)+36421/3\cdot(q-1,7)-13/6\cdot(q-2,7)\\
&\quad-11/2\cdot(q-3,7)-13/6\cdot(q-4,7)-11/2\cdot(q-5,7)+285359/16\cdot(q-1,8)+19\cdot(q-3,8)\\
&\quad+863/16\cdot(q-5,8)+8288/3\cdot(q-1,9)-2\cdot(q-2,9)-10/3\cdot(q-4,9)-2\cdot(q-5,9)\\
&\quad-10/3\cdot(q-7,9)+300\cdot(q-1,11)+35\cdot(q-1,13)+21\cdot(q-1,16)+20976389/8]\cdot q^4\\
&+[-120069\cdot(q,2)\cdot(q-1,3)-7039\cdot(q,2)\cdot(q-1,5)-1747/3\cdot(q,2)\cdot(q-1,7)\\
&\quad-1/3\cdot(q,2)\cdot(q-2,7)-1/3\cdot(q,2)\cdot(q-4,7)+17/2\cdot(q,3)\cdot(q-1,4)\\
&\quad+2851\cdot(q-1,3)\cdot(q-1,4)-(q-1,3)\cdot(q,5)+19\cdot(q-1,3)\cdot(q-1,5)-(q-1,3)\cdot(q-2,5)\\
&\quad-(q-1,3)\cdot(q-3,5)-5515383/8\cdot(q,2)-521\cdot(q,3)+525979\cdot(q-1,3)\\
&\quad+1041755/4\cdot(q-1,4)-53/4\cdot(q,5)+70603\cdot(q-1,5)-13\cdot(q-2,5)-13\cdot(q-3,5)\\
&\quad-7/2\cdot(q,7)+14417\cdot(q-1,7)+1/2\cdot(q-2,7)-7/2\cdot(q-3,7)+1/2\cdot(q-4,7)\\
&\quad-7/2\cdot(q-5,7)+366631/16\cdot(q-1,8)+14\cdot(q-3,8)+1351/16\cdot(q-5,8)+3797\cdot(q-1,9)\\
&\quad-3/2\cdot(q-2,9)-5/2\cdot(q-4,9)-3/2\cdot(q-5,9)-5/2\cdot(q-7,9)+491\cdot(q-1,11)\\
&\quad+80\cdot(q-1,13)+93\cdot(q-1,16)+15200435/8]\cdot q^3\\
&+[-92561\cdot(q,2)\cdot(q-1,3)-6820\cdot(q,2)\cdot(q-1,5)-1/2\cdot(q,2)\cdot(q-2,5)\\
&\quad-1/2\cdot(q,2)\cdot(q-3,5)-727\cdot(q,2)\cdot(q-1,7)-4\cdot(q,3)\cdot(q-1,4)\\
&\quad+3257\cdot(q-1,3)\cdot(q-1,4)-3/2\cdot(q-1,3)\cdot(q,5)+21\cdot(q-1,3)\cdot(q-1,5)\\
&\quad-3/2\cdot(q-1,3)\cdot(q-2,5)-3/2\cdot(q-1,3)\cdot(q-3,5)-3318901/8\cdot(q,2)-621\cdot(q,3)\\
&\quad+724105/2\cdot(q-1,3)+697591/4\cdot(q-1,4)-41/4\cdot(q,5)+52190\cdot(q-1,5)\\
&\quad-19/2\cdot(q-2,5)-19/2\cdot(q-3,5)-7/2\cdot(q,7)+34504/3\cdot(q-1,7)-13/6\cdot(q-2,7)\\
&\quad-7/2\cdot(q-3,7)-13/6\cdot(q-4,7)-7/2\cdot(q-5,7)+318181/16\cdot(q-1,8)+47/2\cdot(q-3,8)\\
&\quad+1981/16\cdot(q-5,8)+3175\cdot(q-1,9)-3/2\cdot(q-2,9)-5/2\cdot(q-4,9)-3/2\cdot(q-5,9)\\
&\quad-5/2\cdot(q-7,9)+454\cdot(q-1,11)+90\cdot(q-1,13)+153\cdot(q-1,16)+9142269/8]\cdot q^2\\
&+[-44208\cdot(q,2)\cdot(q-1,3)-3586\cdot(q,2)\cdot(q-1,5)-447\cdot(q,2)\cdot(q-1,7)\\
&\quad-8\cdot(q,3)\cdot(q-1,4)+1898\cdot(q-1,3)\cdot(q-1,4)-1/2\cdot(q-1,3)\cdot(q,5)\\
&\quad+12\cdot(q-1,3)\cdot(q-1,5)-1/2\cdot(q-1,3)\cdot(q-2,5)-1/2\cdot(q-1,3)\cdot(q-3,5)\\
&\quad-1443055/8\cdot(q,2)-382\cdot(q,3)+320477/2\cdot(q-1,3)+307137/4\cdot(q-1,4)-7/2\cdot(q,5)\\
&\quad+23392\cdot(q-1,5)-3\cdot(q-2,5)-3\cdot(q-3,5)-(q,7)+5360\cdot(q-1,7)+(q-2,7)-(q-3,7)\\
&\quad+(q-4,7)-(q-5,7)+163231/16\cdot(q-1,8)+15/2\cdot(q-3,8)+2231/16\cdot(q-5,8)\\
&\quad+4397/3\cdot(q-1,9)-1/2\cdot(q-2,9)-5/6\cdot(q-4,9)-1/2\cdot(q-5,9)-5/6\cdot(q-7,9)
\end{aligned}
$$

$$
\begin{aligned}
&+1111/5\cdot(q-1,11)+1/5\cdot(q-3,11)+1/5\cdot(q-4,11)+1/5\cdot(q-5,11)+1/5\cdot(q-9,11)\\
&+50\cdot(q-1,13)+111\cdot(q-1,16)+3994177/8]\cdot q\\
+\,&[-9592\cdot(q,2)\cdot(q-1,3)-785\cdot(q,2)\cdot(q-1,5)-325/3\cdot(q,2)\cdot(q-1,7)\\
&-1/3\cdot(q,2)\cdot(q-2,7)-1/3\cdot(q,2)\cdot(q-4,7)-3\cdot(q,3)\cdot(q-1,4)\\
&-1/4\cdot(q,3)\cdot(q-2,5)-1/4\cdot(q,3)\cdot(q-3,5)+443\cdot(q-1,3)\cdot(q-1,4)\\
&-3/4\cdot(q-1,3)\cdot(q,5)+2\cdot(q-1,3)\cdot(q-1,5)-(q-1,3)\cdot(q-2,5)-(q-1,3)\cdot(q-3,5)\\
&-339435/8\cdot(q,2)-391/2\cdot(q,3)+132499/4\cdot(q-1,3)+63573/4\cdot(q-1,4)-2\cdot(q,5)\\
&+4670\cdot(q-1,5)-7/4\cdot(q-2,5)-7/4\cdot(q-3,5)-(q,7)+3262/3\cdot(q-1,7)+7/3\cdot(q-2,7)\\
&-(q-3,7)+7/3\cdot(q-4,7)-(q-5,7)+38163/16\cdot(q-1,8)+10\cdot(q-3,8)+2435/16\cdot(q-5,8)\\
&+854/3\cdot(q-1,9)-1/2\cdot(q-2,9)-5/6\cdot(q-4,9)-1/2\cdot(q-5,9)-5/6\cdot(q-7,9)\\
&+224/5\cdot(q-1,11)-1/5\cdot(q-3,11)-1/5\cdot(q-4,11)-1/5\cdot(q-5,11)-1/5\cdot(q-9,11)\\
&+11\cdot(q-1,13)+30\cdot(q-1,16)+945973/8]
\end{aligned}
$$

A.5.6 Dimension $d = 11$

$$
\begin{aligned}
N_{11,5}(q) = {}& q^{90}+q^{89}+3\cdot q^{88}+5\cdot q^{87}+10\cdot q^{86}+16\cdot q^{85}+28\cdot q^{84}+42\cdot q^{83}+68\cdot q^{82}\\
&+100\cdot q^{81}+151\cdot q^{80}+215\cdot q^{79}+312\cdot q^{78}+432\cdot q^{77}+605\cdot q^{76}\\
&+821\cdot q^{75}+1117\cdot q^{74}+1485\cdot q^{73}+1977\cdot q^{72}+2581\cdot q^{71}+3370\cdot q^{70}\\
&+4333\cdot q^{69}+5561\cdot q^{68}+7050\cdot q^{67}+8918\cdot q^{66}+11160\cdot q^{65}+13931\cdot q^{64}\\
&+17234\cdot q^{63}+21257\cdot q^{62}+26016\cdot q^{61}+31747\cdot q^{60}+38470\cdot q^{59}\\
&+46481\cdot q^{58}+55815\cdot q^{57}+66822\cdot q^{56}+79561\cdot q^{55}+94453\cdot q^{54}\\
&+111567\cdot q^{53}+131412\cdot q^{52}+154080\cdot q^{51}+[-2\cdot(q,2)+180163]\cdot q^{50}\\
&+[-5\cdot(q,2)+209782]\cdot q^{49}+[-13\cdot(q,2)+243635]\cdot q^{48}+[-29\cdot(q,2)+281869]\cdot q^{47}\\
&+[-61\cdot(q,2)+325305]\cdot q^{46}+[-116\cdot(q,2)+374135]\cdot q^{45}\\
&+[-216\cdot(q,2)+429312]\cdot q^{44}+[-380\cdot(q,2)+491099]\cdot q^{43}\\
&+[-655\cdot(q,2)+560631]\cdot q^{42}+[-1089\cdot(q,2)+638252]\cdot q^{41}\\
&+[-1777\cdot(q,2)+725336]\cdot q^{40}+[-2823\cdot(q,2)+822383]\cdot q^{39}\\
&+[-4412\cdot(q,2)+931062]\cdot q^{38}+[-6749\cdot(q,2)+1052121]\cdot q^{37}\\
&+[-10173\cdot(q,2)+1187665]\cdot q^{36}+[-15065\cdot(q,2)+1338790]\cdot q^{35}\\
&+[-22013\cdot(q,2)+1508213]\cdot q^{34}+[-31680\cdot(q,2)+1697596]\cdot q^{33}\\
&+[-45029\cdot(q,2)+1910437]\cdot q^{32}+[-63135\cdot(q,2)+2149181]\cdot q^{31}\\
&+[-87496\cdot(q,2)+1/2\cdot(q,3)+(q-1,3)+4836825/2]\cdot q^{30}\\
&+[-119743\cdot(q,2)+4\cdot(q,3)+10\cdot(q-1,3)+2721524]\cdot q^{29}\\
&+[-162070\cdot(q,2)+19/2\cdot(q,3)+39\cdot(q-1,3)+6128823/2]\cdot q^{28}\\
&+[-216820\cdot(q,2)+47/2\cdot(q,3)+128\cdot(q-1,3)+6903307/2]\cdot q^{27}\\
&+[-287029\cdot(q,2)+81/2\cdot(q,3)+344\cdot(q-1,3)+(q-1,4)+7781073/2]\cdot q^{26}\\
&+[-375859\cdot(q,2)+151/2\cdot(q,3)+843\cdot(q-1,3)+8\cdot(q-1,4)+8773635/2]\cdot q^{25}\\
&+[-487288\cdot(q,2)+219/2\cdot(q,3)+1863\cdot(q-1,3)+36\cdot(q-1,4)+9898241/2]\cdot q^{24}\\
&+[-625319\cdot(q,2)+172\cdot(q,3)+3864\cdot(q-1,3)+124\cdot(q-1,4)+5583958]\cdot q^{23}\\
&+[-794828\cdot(q,2)+218\cdot(q,3)+7496\cdot(q-1,3)+355\cdot(q-1,4)+6300653]\cdot q^{22}\\
&+[-1000478\cdot(q,2)+603/2\cdot(q,3)+13857\cdot(q-1,3)+910\cdot(q-1,4)+14210861/2]\cdot q^{21}\\
&+[-1247620\cdot(q,2)+687/2\cdot(q,3)+24382\cdot(q-1,3)+2112\cdot(q-1,4)+16012917/2]\cdot q^{20}\\
&+[-1540825\cdot(q,2)+871/2\cdot(q,3)+41234\cdot(q-1,3)+4554\cdot(q-1,4)+18013943/2]\cdot q^{19}\\
&+[-1884719\cdot(q,2)+451\cdot(q,3)+67032\cdot(q-1,3)+9220\cdot(q-1,4)+3/4\cdot(q,5)+(q-1,5)\\
&\qquad+40440873/4]\cdot q^{18}\\
&+[-(q,2)\cdot(q-1,3)-2281942\cdot(q,2)+537\cdot(q,3)+105344\cdot(q-1,3)+17713\cdot(q-1,4)
\end{aligned}
$$

$$
\begin{aligned}
&\quad +2\cdot(q,5)+14\cdot(q-1,5)+11311012]\cdot q^{17}\\
&+[-12\cdot(q,2)\cdot(q-1,3)-2733719\cdot(q,2)+510\cdot(q,3)+160104\cdot(q-1,3)+32451\cdot(q-1,4)\\
&\quad +2\cdot(q,5)+85\cdot(q-1,5)-3/2\cdot(q-2,5)-3/2\cdot(q-3,5)+12600411]\cdot q^{16}\\
&+[-88\cdot(q,2)\cdot(q-1,3)-3237187\cdot(q,2)+590\cdot(q,3)+236264\cdot(q-1,3)+57011\cdot(q-1,4)\\
&\quad +4\cdot(q,5)+370\cdot(q-1,5)-(q-2,5)-(q-3,5)+13955549]\cdot q^{15}\\
&+[-442\cdot(q,2)\cdot(q-1,3)-3785506\cdot(q,2)+1051/2\cdot(q,3)+338942\cdot(q-1,3)\\
&\quad +96272\cdot(q-1,4)-1/4\cdot(q,5)+1250\cdot(q-1,5)-13/2\cdot(q-2,5)-13/2\cdot(q-3,5)+(q-1,7)\\
&\quad +61374635/4]\cdot q^{14}\\
&+[-1710\cdot(q,2)\cdot(q-1,3)-4364508\cdot(q,2)+1233/2\cdot(q,3)+474481\cdot(q-1,3)\\
&\quad +156670\cdot(q-1,4)+3\cdot(q,5)+3589\cdot(q-1,5)-9/2\cdot(q-2,5)-9/2\cdot(q-3,5)+10\cdot(q-1,7)\\
&\quad +4\cdot(q-1,8)+33420907/2]\cdot q^{13}\\
&+[-5447\cdot(q,2)\cdot(q-1,3)-4952343\cdot(q,2)+519\cdot(q,3)+649590\cdot(q-1,3)\\
&\quad +245877\cdot(q-1,4)-43/4\cdot(q,5)+8961\cdot(q-1,5)-37/2\cdot(q-2,5)-37/2\cdot(q-3,5)\\
&\quad -1/3\cdot(q,7)+193/3\cdot(q-1,7)-1/6\cdot(q-2,7)-1/2\cdot(q-3,7)-1/6\cdot(q-4,7)\\
&\quad -1/2\cdot(q-5,7)+42\cdot(q-1,8)+215813125/12]\cdot q^{12}\\
&+[-14905\cdot(q,2)\cdot(q-1,3)-11030761/2\cdot(q,2)+1197/2\cdot(q,3)+872830\cdot(q-1,3)\\
&\quad +372206\cdot(q-1,4)-9/2\cdot(q,5)+19983\cdot(q-1,5)-12\cdot(q-2,5)-12\cdot(q-3,5)\\
&\quad -1/2\cdot(q,7)+300\cdot(q-1,7)-1/2\cdot(q-2,7)-1/2\cdot(q-3,7)-1/2\cdot(q-4,7)\\
&\quad -1/2\cdot(q-5,7)+987/4\cdot(q-1,8)+3/4\cdot(q-5,8)+19064299]\cdot q^{11}\\
&+[-35793\cdot(q,2)\cdot(q-1,3)-6008101\cdot(q,2)+394\cdot(q,3)+1152650\cdot(q-1,3)\\
&\quad +542553\cdot(q-1,4)-129/4\cdot(q,5)+40176\cdot(q-1,5)-77/2\cdot(q-2,5)-77/2\cdot(q-3,5)\\
&\quad -7/2\cdot(q,7)+3271/3\cdot(q-1,7)-19/6\cdot(q-2,7)-7/2\cdot(q-3,7)-19/6\cdot(q-4,7)\\
&\quad -7/2\cdot(q-5,7)+2005/2\cdot(q-1,8)+1/2\cdot(q-3,8)+2\cdot(q-5,8)+2/3\cdot(q-1,9)\\
&\quad -1/2\cdot(q-2,9)-5/6\cdot(q-4,9)-1/2\cdot(q-5,9)-5/6\cdot(q-7,9)+79310943/4]\cdot q^{10}\\
&+[-76745\cdot(q,2)\cdot(q-1,3)-13\cdot(q,2)\cdot(q-1,5)-6370169\cdot(q,2)+725/2\cdot(q,3)\\
&\quad +1496936\cdot(q-1,3)+759244\cdot(q-1,4)-31/2\cdot(q,5)+73695\cdot(q-1,5)-23\cdot(q-2,5)\\
&\quad -23\cdot(q-3,5)-3\cdot(q,7)+9961/3\cdot(q-1,7)-5/3\cdot(q-2,7)-3\cdot(q-3,7)-5/3\cdot(q-4,7)\\
&\quad -3\cdot(q-5,7)+3303\cdot(q-1,8)+1/2\cdot(q-3,8)+5/2\cdot(q-5,8)+104/3\cdot(q-1,9)\\
&\quad -1/2\cdot(q-2,9)-5/6\cdot(q-4,9)-1/2\cdot(q-5,9)-5/6\cdot(q-7,9)+20122940]\cdot q^{9}\\
&+[-148261\cdot(q,2)\cdot(q-1,3)-199\cdot(q,2)\cdot(q-1,5)+1/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +1/2\cdot(q,2)\cdot(q-3,5)+4\cdot(q-1,3)\cdot(q-1,4)-52239775/8\cdot(q,2)-39/2\cdot(q,3)\\
&\quad +1905639\cdot(q-1,3)+4057213/4\cdot(q-1,4)-217/4\cdot(q,5)+123999\cdot(q-1,5)\\
&\quad -64\cdot(q-2,5)-64\cdot(q-3,5)-10\cdot(q,7)+25964/3\cdot(q-1,7)-28/3\cdot(q-2,7)\\
&\quad -10\cdot(q-3,7)-28/3\cdot(q-4,7)-10\cdot(q-5,7)+148855/16\cdot(q-1,8)+7\cdot(q-3,8)\\
&\quad +39/16\cdot(q-5,8)+911/3\cdot(q-1,9)-2\cdot(q-2,9)-10/3\cdot(q-4,9)-2\cdot(q-5,9)\\
&\quad -10/3\cdot(q-7,9)+5\cdot(q-1,11)+158332803/8]\cdot q^{8}\\
&+[-259077\cdot(q,2)\cdot(q-1,3)-1400\cdot(q,2)\cdot(q-1,5)+1/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +1/2\cdot(q,2)\cdot(q-3,5)+7/2\cdot(q,3)\cdot(q-1,4)+112\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -25643261/4\cdot(q,2)-463/2\cdot(q,3)+2361584\cdot(q-1,3)+1282693\cdot(q-1,4)\\
&\quad -101/4\cdot(q,5)+192948\cdot(q-1,5)-69/2\cdot(q-2,5)-69/2\cdot(q-3,5)-7\cdot(q,7)\\
&\quad +58592/3\cdot(q-1,7)-16/3\cdot(q-2,7)-7\cdot(q-3,7)-16/3\cdot(q-4,7)-7\cdot(q-5,7)\\
&\quad +181949/8\cdot(q-1,8)+5\cdot(q-3,8)+29/8\cdot(q-5,8)+4652/3\cdot(q-1,9)-2\cdot(q-2,9)\\
&\quad -10/3\cdot(q-4,9)-2\cdot(q-5,9)-10/3\cdot(q-7,9)+65\cdot(q-1,11)+37366967/2]\cdot q^{7}\\
&+[-407423\cdot(q,2)\cdot(q-1,3)-5990\cdot(q,2)\cdot(q-1,5)+1/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +1/2\cdot(q,2)\cdot(q-3,5)-80\cdot(q,2)\cdot(q-1,7)+37/2\cdot(q,3)\cdot(q-1,4)\\
&\quad -1/4\cdot(q,3)\cdot(q-2,5)-1/4\cdot(q,3)\cdot(q-3,5)+1025\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -3/4\cdot(q-1,3)\cdot(q,5)+3\cdot(q-1,3)\cdot(q-1,5)-(q-1,3)\cdot(q-2,5)-(q-1,3)\cdot(q-3,5)\\
&\quad -47587729/8\cdot(q,2)-1493/2\cdot(q,3)+11234447/4\cdot(q-1,3)+6059677/4\cdot(q-1,4)\\
&\quad -147/2\cdot(q,5)+278134\cdot(q-1,5)-305/4\cdot(q-2,5)-305/4\cdot(q-3,5)-17\cdot(q,7)\\
&\quad +114056/3\cdot(q-1,7)-40/3\cdot(q-2,7)-17\cdot(q-3,7)-40/3\cdot(q-4,7)-17\cdot(q-5,7)\\
&\quad +769353/16\cdot(q-1,8)+55/2\cdot(q-3,8)+113/16\cdot(q-5,8)+5297\cdot(q-1,9)-9/2\cdot(q-2,9)
\end{aligned}
$$

$$-15/2 \cdot (q-4,9) - 9/2 \cdot (q-5,9) - 15/2 \cdot (q-7,9) + 388 \cdot (q-1,11) + 15 \cdot (q-1,13)$$
$$+133737163/8] \cdot q^6$$
$$+ [-565936 \cdot (q,2) \cdot (q-1,3) - 17208 \cdot (q,2) \cdot (q-1,5) - 659 \cdot (q,2) \cdot (q-1,7)$$
$$+40 \cdot (q,3) \cdot (q-1,4) + 4660 \cdot (q-1,3) \cdot (q-1,4) - (q-1,3) \cdot (q,5) + 29 \cdot (q-1,3) \cdot (q-1,5)$$
$$-(q-1,3) \cdot (q-2,5) - (q-1,3) \cdot (q-3,5) - 40966473/8 \cdot (q,2) - 1113 \cdot (q,3)$$
$$+3123490 \cdot (q-1,3) + 6538295/4 \cdot (q-1,4) - 129/4 \cdot (q,5) + 366517 \cdot (q-1,5)$$
$$-63/2 \cdot (q-2,5) - 63/2 \cdot (q-3,5) - 8 \cdot (q,7) + 62942 \cdot (q-1,7) - 6 \cdot (q-2,7) - 8 \cdot (q-3,7)$$
$$-6 \cdot (q-4,7) - 8 \cdot (q-5,7) + 1372793/16 \cdot (q-1,8) + 16 \cdot (q-3,8) + 713/16 \cdot (q-5,8)$$
$$+12734 \cdot (q-1,9) - 3 \cdot (q-2,9) - 5 \cdot (q-4,9) - 3 \cdot (q-5,9) - 5 \cdot (q-7,9) + 1339 \cdot (q-1,11)$$
$$+120 \cdot (q-1,13) + 63 \cdot (q-1,16) + 111542461/8] \cdot q^5$$
$$+ [-667181 \cdot (q,2) \cdot (q-1,3) - 33780 \cdot (q,2) \cdot (q-1,5) - 1/2 \cdot (q,2) \cdot (q-2,5)$$
$$-1/2 \cdot (q,2) \cdot (q-3,5) - 2265 \cdot (q,2) \cdot (q-1,7) + 41 \cdot (q,3) \cdot (q-1,4)$$
$$-1/4 \cdot (q,3) \cdot (q-2,5) - 1/4 \cdot (q,3) \cdot (q-3,5) + 12230 \cdot (q-1,3) \cdot (q-1,4)$$
$$-15/4 \cdot (q-1,3) \cdot (q,5) + 94 \cdot (q-1,3) \cdot (q-1,5) - 4 \cdot (q-1,3) \cdot (q-2,5)$$
$$-4 \cdot (q-1,3) \cdot (q-3,5) - 15937249/4 \cdot (q,2) - 1642 \cdot (q,3) + 12410919/4 \cdot (q-1,3)$$
$$+3108265/2 \cdot (q-1,4) - 251/4 \cdot (q,5) + 421343 \cdot (q-1,5) - 243/4 \cdot (q-2,5)$$
$$-243/4 \cdot (q-3,5) - 41/2 \cdot (q,7) + 85017 \cdot (q-1,7) - 35/2 \cdot (q-2,7) - 41/2 \cdot (q-3,7)$$
$$-35/2 \cdot (q-4,7) - 41/2 \cdot (q-5,7) + 988785/8 \cdot (q-1,8) + 103/2 \cdot (q-3,8) + 917/8 \cdot (q-5,8)$$
$$+21467 \cdot (q-1,9) - 15/2 \cdot (q-2,9) - 25/2 \cdot (q-4,9) - 15/2 \cdot (q-5,9) - 25/2 \cdot (q-7,9)$$
$$+2852 \cdot (q-1,11) + 391 \cdot (q-1,13) + 378 \cdot (q-1,16) + 10603089] \cdot q^4$$
$$+ [-625603 \cdot (q,2) \cdot (q-1,3) - 44273 \cdot (q,2) \cdot (q-1,5) - 1/2 \cdot (q,2) \cdot (q-2,5)$$
$$-1/2 \cdot (q,2) \cdot (q-3,5) - 12388/3 \cdot (q,2) \cdot (q-1,7) - 1/3 \cdot (q,2) \cdot (q-2,7)$$
$$-1/3 \cdot (q,2) \cdot (q-4,7) + 17/2 \cdot (q,3) \cdot (q-1,4) + 19367 \cdot (q-1,3) \cdot (q-1,4)$$
$$-2 \cdot (q-1,3) \cdot (q,5) + 164 \cdot (q-1,3) \cdot (q-1,5) - 2 \cdot (q-1,3) \cdot (q-2,5)$$
$$-2 \cdot (q-1,3) \cdot (q-3,5) - 5388647/2 \cdot (q,2) - 1813 \cdot (q,3) + 2563157 \cdot (q-1,3)$$
$$+2459157/2 \cdot (q-1,4) - 65/4 \cdot (q,5) + 389163 \cdot (q-1,5) - 15 \cdot (q-2,5) - 15 \cdot (q-3,5)$$
$$-9/2 \cdot (q,7) + 87219 \cdot (q-1,7) + 5/2 \cdot (q-2,7) - 9/2 \cdot (q-3,7) + 5/2 \cdot (q-4,7)$$
$$-9/2 \cdot (q-5,7) + 538195/4 \cdot (q-1,8) + 43/2 \cdot (q-3,8) + 949/4 \cdot (q-5,8) + 73844/3 \cdot (q-1,9)$$
$$-2 \cdot (q-2,9) - 10/3 \cdot (q-4,9) - 2 \cdot (q-5,9) - 10/3 \cdot (q-7,9) + 3783 \cdot (q-1,11)$$
$$+664 \cdot (q-1,13) + 912 \cdot (q-1,16) + 28393445/4] \cdot q^3$$
$$+ [-424767 \cdot (q,2) \cdot (q-1,3) - 36647 \cdot (q,2) \cdot (q-1,5) - 1/2 \cdot (q,2) \cdot (q-2,5)$$
$$-1/2 \cdot (q,2) \cdot (q-3,5) - 4188 \cdot (q,2) \cdot (q-1,7) - 22 \cdot (q,3) \cdot (q-1,4)$$
$$-1/4 \cdot (q,3) \cdot (q-2,5) - 1/4 \cdot (q,3) \cdot (q-3,5) + 18177 \cdot (q-1,3) \cdot (q-1,4)$$
$$-21/4 \cdot (q-1,3) \cdot (q,5) + 150 \cdot (q-1,3) \cdot (q-1,5) - 11/2 \cdot (q-1,3) \cdot (q-2,5)$$
$$-11/2 \cdot (q-1,3) \cdot (q-3,5) - 11855543/8 \cdot (q,2) - 1744 \cdot (q,3) + 6360665/4 \cdot (q-1,3)$$
$$+2965381/4 \cdot (q-1,4) - 33 \cdot (q,5) + 258059 \cdot (q-1,5) - 127/4 \cdot (q-2,5) - 127/4 \cdot (q-3,5)$$
$$-31/2 \cdot (q,7) + 184511/3 \cdot (q-1,7) - 83/6 \cdot (q-2,7) - 31/2 \cdot (q-3,7) - 83/6 \cdot (q-4,7)$$
$$-31/2 \cdot (q-5,7) + 1618751/16 \cdot (q-1,8) + 95/2 \cdot (q-3,8) + 5703/16 \cdot (q-5,8)$$
$$+54109/3 \cdot (q-1,9) - 7 \cdot (q-2,9) - 35/3 \cdot (q-4,9) - 7 \cdot (q-5,9) - 35/3 \cdot (q-7,9)$$
$$+3032 \cdot (q-1,11) + 621 \cdot (q-1,13) + 1098 \cdot (q-1,16) + 31344385/8] \cdot q^2$$
$$+ [-181056 \cdot (q,2) \cdot (q-1,3) - 17174 \cdot (q,2) \cdot (q-1,5) - 2233 \cdot (q,2) \cdot (q-1,7)$$
$$-23 \cdot (q,3) \cdot (q-1,4) + 9265 \cdot (q-1,3) \cdot (q-1,4) - (q-1,3) \cdot (q,5)$$
$$+75 \cdot (q-1,3) \cdot (q-1,5) - (q-1,3) \cdot (q-2,5) - (q-1,3) \cdot (q-3,5) - 4677279/8 \cdot (q,2)$$
$$-1935/2 \cdot (q,3) + 636798 \cdot (q-1,3) + 1178405/4 \cdot (q-1,4) - 13/4 \cdot (q,5)$$
$$+105409 \cdot (q-1,5) - 7/2 \cdot (q-2,5) - 7/2 \cdot (q-3,5) - (q,7) + 77758/3 \cdot (q-1,7)$$
$$+1/3 \cdot (q-2,7) - (q-3,7) + 1/3 \cdot (q-4,7) - (q-5,7) + 736431/16 \cdot (q-1,8) + 10 \cdot (q-3,8)$$
$$+7135/16 \cdot (q-5,8) + 22694/3 \cdot (q-1,9) - 1/2 \cdot (q-2,9) - 5/6 \cdot (q-4,9) - 1/2 \cdot (q-5,9)$$
$$-5/6 \cdot (q-7,9) + 1341 \cdot (q-1,11) + 304 \cdot (q-1,13) + 657 \cdot (q-1,16) + 12502459/8] \cdot q$$
$$+ [-35640 \cdot (q,2) \cdot (q-1,3) - 3452 \cdot (q,2) \cdot (q-1,5) - 1465/3 \cdot (q,2) \cdot (q-1,7)$$
$$-1/3 \cdot (q,2) \cdot (q-2,7) - 1/3 \cdot (q,2) \cdot (q-4,7) - 17/2 \cdot (q,3) \cdot (q-1,4)$$
$$-1/4 \cdot (q,3) \cdot (q-2,5) - 1/4 \cdot (q,3) \cdot (q-3,5) + 1968 \cdot (q-1,3) \cdot (q-1,4)$$
$$-9/4 \cdot (q-1,3) \cdot (q,5) + 13 \cdot (q-1,3) \cdot (q-1,5) - 5/2 \cdot (q-1,3) \cdot (q-2,5)$$

$$-5/2\cdot(q-1,3)\cdot(q-3,5)-990527/8\cdot(q,2)-453\cdot(q,3)+481285/4\cdot(q-1,3)$$
$$+221651/4\cdot(q-1,4)-17/2\cdot(q,5)+19494\cdot(q-1,5)-33/4\cdot(q-2,5)-33/4\cdot(q-3,5)$$
$$-5\cdot(q,7)+14531/3\cdot(q-1,7)-1/3\cdot(q-2,7)-5\cdot(q-3,7)-1/3\cdot(q-4,7)-5\cdot(q-5,7)$$
$$+155111/16\cdot(q-1,8)+17\cdot(q-3,8)+7911/16\cdot(q-5,8)+4102/3\cdot(q-1,9)-5/2\cdot(q-2,9)$$
$$-25/6\cdot(q-4,9)-5/2\cdot(q-5,9)-25/6\cdot(q-7,9)+251\cdot(q-1,11)+61\cdot(q-1,13)$$
$$+156\cdot(q-1,16)+2685705/8]$$

A.5.7 Dimension $d = 12$

$$
\begin{aligned}
N_{12,5}(q) = {} & q^{102}+q^{101}+3\cdot q^{100}+5\cdot q^{99}+10\cdot q^{98}+16\cdot q^{97}+28\cdot q^{96}+43\cdot q^{95}+69\cdot q^{94} \\
& +103\cdot q^{93}+156\cdot q^{92}+225\cdot q^{91}+328\cdot q^{90}+460\cdot q^{89}+648\cdot q^{88} \\
& +890\cdot q^{87}+1220\cdot q^{86}+1641\cdot q^{85}+2202\cdot q^{84}+2908\cdot q^{83}+3829\cdot q^{82} \\
& +4978\cdot q^{81}+6446\cdot q^{80}+8260\cdot q^{79}+10543\cdot q^{78}+13334\cdot q^{77} \\
& +16797\cdot q^{76}+20995\cdot q^{75}+26135\cdot q^{74}+32311\cdot q^{73}+39792\cdot q^{72} \\
& +48701\cdot q^{71}+59383\cdot q^{70}+72007\cdot q^{69}+86996\cdot q^{68}+104579\cdot q^{67} \\
& +125279\cdot q^{66}+149382\cdot q^{65}+177532\cdot q^{64}+210092\cdot q^{63}+247829\cdot q^{62} \\
& +291197\cdot q^{61}+341114\cdot q^{60}+398121\cdot q^{59}+463307\cdot q^{58}+537324\cdot q^{57} \\
& +[-(q,2)+621432]\cdot q^{56}+[-3\cdot(q,2)+716410]\cdot q^{55}+[-10\cdot(q,2)+823726]\cdot q^{54} \\
& +[-24\cdot(q,2)+944278]\cdot q^{53}+[-56\cdot(q,2)+1079776]\cdot q^{52} \\
& +[-115\cdot(q,2)+1231273]\cdot q^{51}+[-229\cdot(q,2)+1400736]\cdot q^{50} \\
& +[-424\cdot(q,2)+1589410]\cdot q^{49}+[-766\cdot(q,2)+1799601]\cdot q^{48} \\
& +[-1322\cdot(q,2)+2032771]\cdot q^{47}+[-2230\cdot(q,2)+2291656]\cdot q^{46} \\
& +[-3645\cdot(q,2)+2578057]\cdot q^{45}+[-5838\cdot(q,2)+2895237]\cdot q^{44} \\
& +[-9126\cdot(q,2)+3245482]\cdot q^{43}+[-14019\cdot(q,2)+3632825]\cdot q^{42} \\
& +[-21113\cdot(q,2)+4060201]\cdot q^{41}+[-31309\cdot(q,2)+4532702]\cdot q^{40} \\
& +[-45674\cdot(q,2)+5054289]\cdot q^{39}+[-65721\cdot(q,2)+5631454]\cdot q^{38} \\
& +[-93233\cdot(q,2)+6269607]\cdot q^{37}+[-130649\cdot(q,2)+6977228]\cdot q^{36} \\
& +[-180786\cdot(q,2)+7761630]\cdot q^{35} \\
& +[-247372\cdot(q,2)+1/2\cdot(q,3)+(q-1,3)+17267801/2]\cdot q^{34} \\
& +[-334651\cdot(q,2)+4\cdot(q,3)+10\cdot(q-1,3)+9603918]\cdot q^{33} \\
& +[-448034\cdot(q,2)+10\cdot(q,3)+40\cdot(q-1,3)+10685894]\cdot q^{32} \\
& +[-593576\cdot(q,2)+53/2\cdot(q,3)+137\cdot(q-1,3)+23785587/2]\cdot q^{31} \\
& +[-778786\cdot(q,2)+49\cdot(q,3)+381\cdot(q-1,3)+13242551]\cdot q^{30} \\
& +[-1011846\cdot(q,2)+189/2\cdot(q,3)+964\cdot(q-1,3)+2\cdot(q-1,4)+29502671/2]\cdot q^{29} \\
& +[-1302646\cdot(q,2)+299/2\cdot(q,3)+2200\cdot(q-1,3)+15\cdot(q-1,4)+32881743/2]\cdot q^{28} \\
& +[-1661721\cdot(q,2)+483/2\cdot(q,3)+4697\cdot(q-1,3)+61\cdot(q-1,4)+36660693/2]\cdot q^{27} \\
& +[-2101417\cdot(q,2)+673/2\cdot(q,3)+9394\cdot(q-1,3)+207\cdot(q-1,4)+40888841/2]\cdot q^{26} \\
& +[-2634516\cdot(q,2)+479\cdot(q,3)+17873\cdot(q-1,3)+593\cdot(q-1,4)+22803858]\cdot q^{25} \\
& +[-3275541\cdot(q,2)+1205/2\cdot(q,3)+32413\cdot(q-1,3)+1519\cdot(q-1,4)+50868437/2]\cdot q^{24} \\
& +[-4038734\cdot(q,2)+1557/2\cdot(q,3)+56483\cdot(q-1,3)+3548\cdot(q-1,4)+56708119/2]\cdot q^{23} \\
& +[-4939502\cdot(q,2)+899\cdot(q,3)+94743\cdot(q-1,3)+7707\cdot(q-1,4)+31584736]\cdot q^{22} \\
& +[-5991410\cdot(q,2)+2157/2\cdot(q,3)+153688\cdot(q-1,3)+15745\cdot(q-1,4)+70270535/2]\cdot q^{21} \\
& +[-7207405\cdot(q,2)+2313/2\cdot(q,3)+241428\cdot(q-1,3)+30518\cdot(q-1,4)+3/4\cdot(q,5) \\
& \quad +3\cdot(q-1,5)-1/2\cdot(q-2,5)-1/2\cdot(q-3,5)+156048999/4]\cdot q^{20} \\
& +[-(q,2)\cdot(q-1,3)-8595488\cdot(q,2)+2617/2\cdot(q,3)+368343\cdot(q-1,3)+56485\cdot(q-1,4)
\end{aligned}
$$

$$
\begin{aligned}
&+11/4\cdot(q,5)+33\cdot(q-1,5)-1/2\cdot(q-2,5)-1/2\cdot(q-3,5)+172804335/4]\cdot q^{19}\\
&+[-13\cdot(q,2)\cdot(q-1,3)-10159622\cdot(q,2)+2623/2\cdot(q,3)+546419\cdot(q-1,3)\\
&\quad+100334\cdot(q-1,4)+15/4\cdot(q,5)+183\cdot(q-1,5)-5/2\cdot(q-2,5)-5/2\cdot(q-3,5)\\
&\quad+190703663/4]\cdot q^{18}\\
&+[-106\cdot(q,2)\cdot(q-1,3)-11893324\cdot(q,2)+2853/2\cdot(q,3)+789788\cdot(q-1,3)\\
&\quad+171667\cdot(q-1,4)+7\cdot(q,5)+756\cdot(q-1,5)-5/2\cdot(q-2,5)-5/2\cdot(q-3,5)+104750437/2]\cdot q^{17}\\
&+[-575\cdot(q,2)\cdot(q-1,3)-13780362\cdot(q,2)+1362\cdot(q,3)+1113651\cdot(q-1,3)\\
&\quad+283685\cdot(q-1,4)+7/2\cdot(q,5)+2520\cdot(q-1,5)-9\cdot(q-2,5)-9\cdot(q-3,5)+(q-1,7)\\
&\quad+114425447/2]\cdot q^{16}\\
&+[-2401\cdot(q,2)\cdot(q-1,3)-15786061\cdot(q,2)+2957/2\cdot(q,3)+1535226\cdot(q-1,3)\\
&\quad+453726\cdot(q-1,4)+13/2\cdot(q,5)+7215\cdot(q-1,5)-17/2\cdot(q-2,5)-17/2\cdot(q-3,5)\\
&\quad+34/3\cdot(q-1,7)+1/3\cdot(q-2,7)+1/3\cdot(q-4,7)+2\cdot(q-1,8)+62047394]\cdot q^{15}\\
&+[-8150\cdot(q,2)\cdot(q-1,3)-17857608\cdot(q,2)+2737/2\cdot(q,3)+2072907\cdot(q-1,3)\\
&\quad+703205\cdot(q-1,4)-29/4\cdot(q,5)+18187\cdot(q-1,5)-24\cdot(q-2,5)-24\cdot(q-3,5)\\
&\quad-1/6\cdot(q,7)+220/3\cdot(q-1,7)-1/6\cdot(q-2,7)-1/2\cdot(q-3,7)-1/6\cdot(q-4,7)\\
&\quad-1/2\cdot(q-5,7)+32\cdot(q-1,8)+800346671/12]\cdot q^{14}\\
&+[-23598\cdot(q,2)\cdot(q-1,3)-19913315\cdot(q,2)+1487\cdot(q,3)+2748400\cdot(q-1,3)\\
&\quad+1056782\cdot(q-1,4)-9/2\cdot(q,5)+41295\cdot(q-1,5)-22\cdot(q-2,5)-22\cdot(q-3,5)\\
&\quad-1/6\cdot(q,7)+1067/3\cdot(q-1,7)+1/6\cdot(q-2,7)-1/2\cdot(q-3,7)+1/6\cdot(q-4,7)\\
&\quad-1/2\cdot(q-5,7)+449/2\cdot(q-1,8)-1/2\cdot(q-3,8)+212695067/3]\cdot q^{13}\\
&+[-59859\cdot(q,2)\cdot(q-1,3)-21843273\cdot(q,2)+2497/2\cdot(q,3)+3585677\cdot(q-1,3)\\
&\quad+1539538\cdot(q-1,4)-143/4\cdot(q,5)+85371\cdot(q-1,5)-105/2\cdot(q-2,5)-105/2\cdot(q-3,5)\\
&\quad-10/3\cdot(q,7)+4091/3\cdot(q-1,7)-11/6\cdot(q-2,7)-7/2\cdot(q-3,7)-11/6\cdot(q-4,7)\\
&\quad-7/2\cdot(q-5,7)+2145/2\cdot(q-1,8)-1/2\cdot(q-3,8)+(q-5,8)+892083331/12]\cdot q^{12}\\
&+[-135663\cdot(q,2)\cdot(q-1,3)-(q,2)\cdot(q-1,5)-46996085/2\cdot(q,2)+2427/2\cdot(q,3)\\
&\quad+4611318\cdot(q-1,3)+2171517\cdot(q-1,4)-119/4\cdot(q,5)+162247\cdot(q-1,5)-46\cdot(q-2,5)\\
&\quad-46\cdot(q-3,5)-7/2\cdot(q,7)+13166/3\cdot(q-1,7)-11/6\cdot(q-2,7)-7/2\cdot(q-3,7)\\
&\quad-11/6\cdot(q-4,7)-7/2\cdot(q-5,7)+15607/4\cdot(q-1,8)-2\cdot(q-3,8)+7/4\cdot(q-5,8)\\
&\quad+6\cdot(q-1,9)+306483405/4]\cdot q^{11}\\
&+[-278061\cdot(q,2)\cdot(q-1,3)-33\cdot(q,2)\cdot(q-1,5)+1/2\cdot(q,2)\cdot(q-2,5)\\
&\quad+1/2\cdot(q,2)\cdot(q-3,5)-49390105/2\cdot(q,2)+630\cdot(q,3)+5846242\cdot(q-1,3)\\
&\quad+2957781\cdot(q-1,4)-309/4\cdot(q,5)+284942\cdot(q-1,5)-189/2\cdot(q-2,5)\\
&\quad-189/2\cdot(q-3,5)-25/2\cdot(q,7)+36563/3\cdot(q-1,7)-59/6\cdot(q-2,7)-25/2\cdot(q-3,7)\\
&\quad-59/6\cdot(q-4,7)-25/2\cdot(q-5,7)+47207/4\cdot(q-1,8)-1/4\cdot(q-5,8)+96\cdot(q-1,9)\\
&\quad-3/2\cdot(q-2,9)-5/2\cdot(q-4,9)-3/2\cdot(q-5,9)-5/2\cdot(q-7,9)+309186813/4]\cdot q^{10}\\
&+[-519937\cdot(q,2)\cdot(q-1,3)-448\cdot(q,2)\cdot(q-1,5)+2\cdot(q,3)\cdot(q-1,4)\\
&\quad+12\cdot(q-1,3)\cdot(q-1,4)-201733445/8\cdot(q,2)+523/2\cdot(q,3)+7293574\cdot(q-1,3)\\
&\quad+15497239/4\cdot(q-1,4)-223/4\cdot(q,5)+464983\cdot(q-1,5)-77\cdot(q-2,5)-77\cdot(q-3,5)\\
&\quad-12\cdot(q,7)+89599/3\cdot(q-1,7)-23/3\cdot(q-2,7)-12\cdot(q-3,7)-23/3\cdot(q-4,7)\\
&\quad-12\cdot(q-5,7)+496613/16\cdot(q-1,8)-1/2\cdot(q-3,8)-275/16\cdot(q-5,8)+772\cdot(q-1,9)\\
&\quad-3/2\cdot(q-2,9)-5/2\cdot(q-4,9)-3/2\cdot(q-5,9)-5/2\cdot(q-7,9)+10\cdot(q-1,11)\\
&\quad+607054297/8]\cdot q^{9}\\
&+[-890749\cdot(q,2)\cdot(q-1,3)-3119\cdot(q,2)\cdot(q-1,5)+5/2\cdot(q,2)\cdot(q-2,5)\\
&\quad+5/2\cdot(q,2)\cdot(q-3,5)+13/2\cdot(q,3)\cdot(q-1,4)+232\cdot(q-1,3)\cdot(q-1,4)\\
&\quad-24837999\cdot(q,2)-1475/2\cdot(q,3)+8909283\cdot(q-1,3)+9699505/2\cdot(q-1,4)\\
&\quad-115\cdot(q,5)+708354\cdot(q-1,5)-279/2\cdot(q-2,5)-279/2\cdot(q-3,5)-28\cdot(q,7)\\
&\quad+195103/3\cdot(q-1,7)-71/3\cdot(q-2,7)-28\cdot(q-3,7)-71/3\cdot(q-4,7)-28\cdot(q-5,7)\\
&\quad+145071/2\cdot(q-1,8)+13\cdot(q-3,8)-111/2\cdot(q-5,8)+11578/3\cdot(q-1,9)-7\cdot(q-2,9)\\
&\quad-35/3\cdot(q-4,9)-7\cdot(q-5,9)-35/3\cdot(q-7,9)+133\cdot(q-1,11)+143897669/2]\cdot q^{8}\\
&+[-1396854\cdot(q,2)\cdot(q-1,3)-13924\cdot(q,2)\cdot(q-1,5)+1/2\cdot(q,2)\cdot(q-2,5)\\
&\quad+1/2\cdot(q,2)\cdot(q-3,5)-107\cdot(q,2)\cdot(q-1,7)+71/2\cdot(q,3)\cdot(q-1,4)
\end{aligned}
$$

$$
\begin{aligned}
&\quad +2059\cdot(q-1,3)\cdot(q-1,4)-(q-1,3)\cdot(q,5)+4\cdot(q-1,3)\cdot(q-1,5)-(q-1,3)\cdot(q-2,5)\\
&\quad -(q-1,3)\cdot(q-3,5)-186878885/8\cdot(q,2)-2809/2\cdot(q,3)+10558491\cdot(q-1,3)\\
&\quad +22989149/4\cdot(q-1,4)-85\cdot(q,5)+1011545\cdot(q-1,5)-101\cdot(q-2,5)-101\cdot(q-3,5)\\
&\quad -25\cdot(q,7)+377417/3\cdot(q-1,7)-55/3\cdot(q-2,7)-25\cdot(q-3,7)-55/3\cdot(q-4,7)\\
&\quad -25\cdot(q-5,7)+2415365/16\cdot(q-1,8)+31/2\cdot(q-3,8)-1587/16\cdot(q-5,8)\\
&\quad +41176/3\cdot(q-1,9)-7\cdot(q-2,9)-35/3\cdot(q-4,9)-7\cdot(q-5,9)-35/3\cdot(q-7,9)\\
&\quad +851\cdot(q-1,11)+20\cdot(q-1,13)+521743987/8]\cdot q^7\\
&+[-1985509\cdot(q,2)\cdot(q-1,3)-43630\cdot(q,2)\cdot(q-1,5)+5/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +5/2\cdot(q,2)\cdot(q-3,5)-3344/3\cdot(q,2)\cdot(q-1,7)-2/3\cdot(q,2)\cdot(q-2,7)\\
&\quad -2/3\cdot(q,2)\cdot(q-4,7)+80\cdot(q,3)\cdot(q-1,4)-1/4\cdot(q,3)\cdot(q-2,5)\\
&\quad -1/4\cdot(q,3)\cdot(q-3,5)+10143\cdot(q-1,3)\cdot(q-1,4)-7/4\cdot(q-1,3)\cdot(q,5)\\
&\quad +52\cdot(q-1,3)\cdot(q-1,5)-2\cdot(q-1,3)\cdot(q-2,5)-2\cdot(q-1,3)\cdot(q-3,5)\\
&\quad -82749301/4\cdot(q,2)-2618\cdot(q,3)+47762663/4\cdot(q-1,3)+12708533/2\cdot(q-1,4)\\
&\quad -277/2\cdot(q,5)+1349040\cdot(q-1,5)-579/4\cdot(q-2,5)-579/4\cdot(q-3,5)-79/2\cdot(q,7)\\
&\quad +642844/3\cdot(q-1,7)-169/6\cdot(q-2,7)-79/2\cdot(q-3,7)-169/6\cdot(q-4,7)\\
&\quad -79/2\cdot(q-5,7)+2207633/8\cdot(q-1,8)+54\cdot(q-3,8)-799/8\cdot(q-5,8)+36155\cdot(q-1,9)\\
&\quad -27/2\cdot(q-2,9)-45/2\cdot(q-4,9)-27/2\cdot(q-5,9)-45/2\cdot(q-7,9)+3360\cdot(q-1,11)\\
&\quad +215\cdot(q-1,13)+105\cdot(q-1,16)+223104643/4]\cdot q^6\\
&+[-2499259\cdot(q,2)\cdot(q-1,3)-99069\cdot(q,2)\cdot(q-1,5)-5008\cdot(q,2)\cdot(q-1,7)\\
&\quad +107\cdot(q,3)\cdot(q-1,4)+31375\cdot(q-1,3)\cdot(q-1,4)-5\cdot(q-1,3)\cdot(q,5)\\
&\quad +235\cdot(q-1,3)\cdot(q-1,5)-5\cdot(q-1,3)\cdot(q-2,5)-5\cdot(q-1,3)\cdot(q-3,5)\\
&\quad -33846149/2\cdot(q,2)-6833/2\cdot(q,3)+12523648\cdot(q-1,3)+6402666\cdot(q-1,4)\\
&\quad -359/4\cdot(q,5)+1643507\cdot(q-1,5)-89\cdot(q-2,5)-89\cdot(q-3,5)-61/2\cdot(q,7)\\
&\quad +943127/3\cdot(q-1,7)-149/6\cdot(q-2,7)-61/2\cdot(q-3,7)-149/6\cdot(q-4,7)\\
&\quad -61/2\cdot(q-5,7)+1721305/4\cdot(q-1,8)+95/2\cdot(q-3,8)-5/4\cdot(q-5,8)+71150\cdot(q-1,9)\\
&\quad -12\cdot(q-2,9)-20\cdot(q-4,9)-12\cdot(q-5,9)-20\cdot(q-7,9)+8781\cdot(q-1,11)\\
&\quad +960\cdot(q-1,13)+819\cdot(q-1,16)+176996821/4]\cdot q^5\\
&+[-2671742\cdot(q,2)\cdot(q-1,3)-161832\cdot(q,2)\cdot(q-1,5)-3/2\cdot(q,2)\cdot(q-2,5)\\
&\quad -3/2\cdot(q,2)\cdot(q-3,5)-12489\cdot(q,2)\cdot(q-1,7)+143/2\cdot(q,3)\cdot(q-1,4)\\
&\quad -1/4\cdot(q,3)\cdot(q-2,5)-1/4\cdot(q,3)\cdot(q-3,5)+63463\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -31/4\cdot(q-1,3)\cdot(q,5)+571\cdot(q-1,3)\cdot(q-1,5)-8\cdot(q-1,3)\cdot(q-2,5)\\
&\quad -8\cdot(q-1,3)\cdot(q-3,5)-99632089/8\cdot(q,2)-4397\cdot(q,3)+46508171/4\cdot(q-1,3)\\
&\quad +22661465/4\cdot(q-1,4)-361/4\cdot(q,5)+1738572\cdot(q-1,5)-349/4\cdot(q-2,5)\\
&\quad -349/4\cdot(q-3,5)-71/2\cdot(q,7)+1139099/3\cdot(q-1,7)-161/6\cdot(q-2,7)\\
&\quad -71/2\cdot(q-3,7)-161/6\cdot(q-4,7)-71/2\cdot(q-5,7)+8746489/16\cdot(q-1,8)\\
&\quad +205/2\cdot(q-3,8)+4433/16\cdot(q-5,8)+307867/3\cdot(q-1,9)-29/2\cdot(q-2,9)\\
&\quad -145/6\cdot(q-4,9)-29/2\cdot(q-5,9)-145/6\cdot(q-7,9)+15380\cdot(q-1,11)+2315\cdot(q-1,13)\\
&\quad +2709\cdot(q-1,16)+254844217/8]\cdot q^4\\
&+[-2277223\cdot(q,2)\cdot(q-1,3)-183774\cdot(q,2)\cdot(q-1,5)-1/2\cdot(q,2)\cdot(q-2,5)\\
&\quad -1/2\cdot(q,2)\cdot(q-3,5)-55667/3\cdot(q,2)\cdot(q-1,7)-5/3\cdot(q,2)\cdot(q-2,7)\\
&\quad -5/3\cdot(q,2)\cdot(q-4,7)-37/2\cdot(q,3)\cdot(q-1,4)+83650\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -7\cdot(q-1,3)\cdot(q,5)+808\cdot(q-1,3)\cdot(q-1,5)-7\cdot(q-1,3)\cdot(q-2,5)\\
&\quad -7\cdot(q-1,3)\cdot(q-3,5)-63417941/8\cdot(q,2)-4313\cdot(q,3)+8920739\cdot(q-1,3)\\
&\quad +16639181/4\cdot(q-1,4)-183/4\cdot(q,5)+1475533\cdot(q-1,5)-89/2\cdot(q-2,5)\\
&\quad -89/2\cdot(q-3,5)-39/2\cdot(q,7)+1055432/3\cdot(q-1,7)-17/6\cdot(q-2,7)-39/2\cdot(q-3,7)\\
&\quad -17/6\cdot(q-4,7)-39/2\cdot(q-5,7)+8491005/16\cdot(q-1,8)+119/2\cdot(q-3,8)\\
&\quad +10197/16\cdot(q-5,8)+104174\cdot(q-1,9)-9\cdot(q-2,9)-15\cdot(q-4,9)-9\cdot(q-5,9)\\
&\quad -15\cdot(q-7,9)+17733\cdot(q-1,11)+3260\cdot(q-1,13)+4812\cdot(q-1,16)+160844153/8]\cdot q^3\\
&+[-1414088\cdot(q,2)\cdot(q-1,3)-136222\cdot(q,2)\cdot(q-1,5)-3\cdot(q,2)\cdot(q-2,5)\\
&\quad -3\cdot(q,2)\cdot(q-3,5)-16340\cdot(q,2)\cdot(q-1,7)-68\cdot(q,3)\cdot(q-1,4)\\
&\quad -1/4\cdot(q,3)\cdot(q-2,5)-1/4\cdot(q,3)\cdot(q-3,5)+68837\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -41/4\cdot(q-1,3)\cdot(q,5)+661\cdot(q-1,3)\cdot(q-1,5)-21/2\cdot(q-1,3)\cdot(q-2,5)
\end{aligned}
$$

$$
\begin{aligned}
&-21/2\cdot(q-1,3)\cdot(q-3,5)-4085047\cdot(q,2)-7359/2\cdot(q,3)+20536785/4\cdot(q-1,3)\\
&+2325755\cdot(q-1,4)-29\cdot(q,5)+904483\cdot(q-1,5)-101/4\cdot(q-2,5)-101/4\cdot(q-3,5)\\
&-39/2\cdot(q,7)+227246\cdot(q-1,7)-33/2\cdot(q-2,7)-39/2\cdot(q-3,7)-33/2\cdot(q-4,7)\\
&-39/2\cdot(q-5,7)+361067\cdot(q-1,8)+95\cdot(q-3,8)+1044\cdot(q-5,8)+69579\cdot(q-1,9)\\
&-9\cdot(q-2,9)-15\cdot(q-4,9)-9\cdot(q-5,9)-15\cdot(q-7,9)+12836\cdot(q-1,11)\\
&+2685\cdot(q-1,13)+4797\cdot(q-1,16)+10427723]\cdot q^2\\
+&[-556085\cdot(q,2)\cdot(q-1,3)-58684\cdot(q,2)\cdot(q-1,5)-7868\cdot(q,2)\cdot(q-1,7)\\
&-111/2\cdot(q,3)\cdot(q-1,4)+31900\cdot(q-1,3)\cdot(q-1,4)-3\cdot(q-1,3)\cdot(q,5)\\
&+297\cdot(q-1,3)\cdot(q-1,5)-3\cdot(q-1,3)\cdot(q-2,5)-3\cdot(q-1,3)\cdot(q-3,5)\\
&-12027287/8\cdot(q,2)-1895\cdot(q,3)+1913900\cdot(q-1,3)+3434695/4\cdot(q-1,4)\\
&-39/4\cdot(q,5)+345010\cdot(q-1,5)-10\cdot(q-2,5)-10\cdot(q-3,5)-5\cdot(q,7)\\
&+266978/3\cdot(q-1,7)-4/3\cdot(q-2,7)-5\cdot(q-3,7)-4/3\cdot(q-4,7)-5\cdot(q-5,7)\\
&+2414063/16\cdot(q-1,8)+53/2\cdot(q-3,8)+20919/16\cdot(q-5,8)+81526/3\cdot(q-1,9)\\
&-5/2\cdot(q-2,9)-25/6\cdot(q-4,9)-5/2\cdot(q-5,9)-25/6\cdot(q-7,9)+5265\cdot(q-1,11)\\
&+1200\cdot(q-1,13)+2529\cdot(q-1,16)+31197059/8]\cdot q\\
+&[-102064\cdot(q,2)\cdot(q-1,3)-11064\cdot(q,2)\cdot(q-1,5)-(q,2)\cdot(q-2,5)\\
&-(q,2)\cdot(q-3,5)-1595\cdot(q,2)\cdot(q-1,7)-(q,2)\cdot(q-2,7)-(q,2)\cdot(q-4,7)\\
&-39/2\cdot(q,3)\cdot(q-1,4)-1/4\cdot(q,3)\cdot(q-2,5)-1/4\cdot(q,3)\cdot(q-3,5)\\
&+6321\cdot(q-1,3)\cdot(q-1,4)-17/4\cdot(q-1,3)\cdot(q,5)+52\cdot(q-1,3)\cdot(q-1,5)\\
&-9/2\cdot(q-1,3)\cdot(q-2,5)-9/2\cdot(q-1,3)\cdot(q-3,5)-2365607/8\cdot(q,2)\\
&-1635/2\cdot(q,3)+1355289/4\cdot(q-1,3)+600795/4\cdot(q-1,4)-7/2\cdot(q,5)\\
&+60283\cdot(q-1,5)-9/4\cdot(q-2,5)-9/4\cdot(q-3,5)-5\cdot(q,7)+47015/3\cdot(q-1,7)\\
&+11/3\cdot(q-2,7)-5\cdot(q-3,7)+11/3\cdot(q-4,7)-5\cdot(q-5,7)+474143/16\cdot(q-1,8)\\
&+34\cdot(q-3,8)+23471/16\cdot(q-5,8)+13993/3\cdot(q-1,9)-5/2\cdot(q-2,9)-25/6\cdot(q-4,9)\\
&-5/2\cdot(q-5,9)-25/6\cdot(q-7,9)+931\cdot(q-1,11)+225\cdot(q-1,13)+549\cdot(q-1,16)\\
&+6260933/8]
\end{aligned}
$$

A.5.8 Dimension $d = 13$

$$
\begin{aligned}
N_{13,5}(q) = {}& q^{112}+q^{111}+3\cdot q^{110}+5\cdot q^{109}+10\cdot q^{108}+16\cdot q^{107}+28\cdot q^{106}+43\cdot q^{105}+70\cdot q^{104}\\
&+104\cdot q^{103}+159\cdot q^{102}+230\cdot q^{101}+338\cdot q^{100}+476\cdot q^{99}+676\cdot q^{98}\\
&+933\cdot q^{97}+1290\cdot q^{96}+1745\cdot q^{95}+2360\cdot q^{94}+3137\cdot q^{93}+4164\cdot q^{92}\\
&+5449\cdot q^{91}+7112\cdot q^{90}+9177\cdot q^{89}+11805\cdot q^{88}+15036\cdot q^{87}\\
&+19088\cdot q^{86}+24029\cdot q^{85}+30143\cdot q^{84}+37535\cdot q^{83}+46576\cdot q^{82}\\
&+57418\cdot q^{81}+70540\cdot q^{80}+86153\cdot q^{79}+104864\cdot q^{78}+126967\cdot q^{77}\\
&+153220\cdot q^{76}+184009\cdot q^{75}+220279\cdot q^{74}+262532\cdot q^{73}+311923\cdot q^{72}\\
&+369090\cdot q^{71}+435435\cdot q^{70}+511754\cdot q^{69}+599730\cdot q^{68}+700337\cdot q^{67}\\
&+815570\cdot q^{66}+946614\cdot q^{65}+1095809\cdot q^{64}+1264559\cdot q^{63}+1455595\cdot q^{62}\\
&+1670567\cdot q^{61}+[-2\cdot(q,2)+1912623]\cdot q^{60}+[-6\cdot(q,2)+2183680]\cdot q^{59}\\
&+[-18\cdot(q,2)+2487346]\cdot q^{58}+[-44\cdot(q,2)+2825837]\cdot q^{57}\\
&+[-102\cdot(q,2)+3203272]\cdot q^{56}+[-211\cdot(q,2)+3622194]\cdot q^{55}\\
&+[-421\cdot(q,2)+4087301]\cdot q^{54}+[-790\cdot(q,2)+4601550]\cdot q^{53}\\
&+[-1432\cdot(q,2)+5170313]\cdot q^{52}+[-2492\cdot(q,2)+5797047]\cdot q^{51}\\
&+[-4217\cdot(q,2)+6487973]\cdot q^{50}+[-6918\cdot(q,2)+7247235]\cdot q^{49}\\
&+[-11095\cdot(q,2)+8082131]\cdot q^{48}+[-17367\cdot(q,2)+8997755]\cdot q^{47}\\
&+[-26666\cdot(q,2)+10002854]\cdot q^{46}+[-40151\cdot(q,2)+11103895]\cdot q^{45}\\
&+[-59474\cdot(q,2)+12311586]\cdot q^{44}+[-86655\cdot(q,2)+13634346]\cdot q^{43}
\end{aligned}
$$

$$
\begin{aligned}
&+[-124488\cdot(q,2)+15085590]\cdot q^{42}+[-176326\cdot(q,2)+16676491]\cdot q^{41}\\
&+[-246653\cdot(q,2)+18424123]\cdot q^{40}+[-340780\cdot(q,2)+20343458]\cdot q^{39}\\
&+[-465592\cdot(q,2)+22456386]\cdot q^{38}\\
&+[-629096\cdot(q,2)+3/2\cdot(q,3)+3\cdot(q-1,3)+49565831/2]\cdot q^{37}\\
&+[-841414\cdot(q,2)+9/2\cdot(q,3)+15\cdot(q-1,3)+54702137/2]\cdot q^{36}\\
&+[-1114077\cdot(q,2)+31/2\cdot(q,3)+64\cdot(q-1,3)+60374173/2]\cdot q^{35}\\
&+[-1461299\cdot(q,2)+33\cdot(q,3)+201\cdot(q-1,3)+33326403]\cdot q^{34}\\
&+[-1898963\cdot(q,2)+145/2\cdot(q,3)+564\cdot(q-1,3)+73605107/2]\cdot q^{33}\\
&+[-2446157\cdot(q,2)+125\cdot(q,3)+1394\cdot(q-1,3)+2\cdot(q-1,4)+40659121]\cdot q^{32}\\
&+[-3123751\cdot(q,2)+224\cdot(q,3)+3202\cdot(q-1,3)+11\cdot(q-1,4)+44937305]\cdot q^{31}\\
&+[-3956252\cdot(q,2)+675/2\cdot(q,3)+6815\cdot(q-1,3)+50\cdot(q-1,4)+99377477/2]\cdot q^{30}\\
&+[-4969803\cdot(q,2)+523\cdot(q,3)+13739\cdot(q-1,3)+178\cdot(q-1,4)+54961187]\cdot q^{29}\\
&+[-6194375\cdot(q,2)+708\cdot(q,3)+26255\cdot(q-1,3)+537\cdot(q-1,4)+60812438]\cdot q^{28}\\
&+[-7661048\cdot(q,2)+987\cdot(q,3)+48089\cdot(q-1,3)+1440\cdot(q-1,4)+67293495]\cdot q^{27}\\
&+[-9404400\cdot(q,2)+2443/2\cdot(q,3)+84534\cdot(q-1,3)+3515\cdot(q-1,4)+148926143/2]\cdot q^{26}\\
&+[-11458819\cdot(q,2)+3135/2\cdot(q,3)+143498\cdot(q-1,3)+7943\cdot(q-1,4)+164735849/2]\cdot q^{25}\\
&+[-13860769\cdot(q,2)+1795\cdot(q,3)+235536\cdot(q-1,3)+16813\cdot(q-1,4)+91056934]\cdot q^{24}\\
&+[-16643562\cdot(q,2)+4317/2\cdot(q,3)+375254\cdot(q-1,3)+33680\cdot(q-1,4)+201114283/2]\cdot q^{23}\\
&+[-19839054\cdot(q,2)+4625/2\cdot(q,3)+580885\cdot(q-1,3)+64249\cdot(q-1,4)+3/4\cdot(q,5)\\
&\qquad+3\cdot(q-1,5)-1/2\cdot(q-2,5)-1/2\cdot(q-3,5)+443548247/4]\cdot q^{22}\\
&+[-23470181\cdot(q,2)+5281/2\cdot(q,3)+875872\cdot(q-1,3)+117452\cdot(q-1,4)+3\cdot(q,5)\\
&\qquad+35\cdot(q-1,5)-1/2\cdot(q-2,5)-1/2\cdot(q-3,5)+244053309/2]\cdot q^{21}\\
&+[-5\cdot(q,2)\cdot(q-1,3)-27551296\cdot(q,2)+2654\cdot(q,3)+1287448\cdot(q-1,3)\\
&\qquad+206536\cdot(q-1,4)+9/2\cdot(q,5)+201\cdot(q-1,5)-3\cdot(q-2,5)-3\cdot(q-3,5)+267856393/2]\cdot q^{20}\\
&+[-57\cdot(q,2)\cdot(q-1,3)-32078100\cdot(q,2)+2917\cdot(q,3)+1848082\cdot(q-1,3)\\
&\qquad+350682\cdot(q-1,4)+39/4\cdot(q,5)+864\cdot(q-1,5)-3\cdot(q-2,5)-3\cdot(q-3,5)+585920841/4]\cdot q^{19}\\
&+[-384\cdot(q,2)\cdot(q-1,3)-37026099\cdot(q,2)+2775\cdot(q,3)+2592767\cdot(q-1,3)\\
&\qquad+576322\cdot(q-1,4)+27/4\cdot(q,5)+2994\cdot(q-1,5)-23/2\cdot(q-2,5)-23/2\cdot(q-3,5)\\
&\qquad+638059881/4]\cdot q^{18}\\
&+[-1871\cdot(q,2)\cdot(q-1,3)-42337388\cdot(q,2)+3013\cdot(q,3)+3560775\cdot(q-1,3)\\
&\qquad+918807\cdot(q-1,4)+13\cdot(q,5)+8927\cdot(q-1,5)-10\cdot(q-2,5)-10\cdot(q-3,5)+4\cdot(q-1,7)\\
&\qquad+172764968]\cdot q^{17}\\
&+[-7177\cdot(q,2)\cdot(q-1,3)-47917500\cdot(q,2)+2782\cdot(q,3)+4792701\cdot(q-1,3)\\
&\qquad+1422963\cdot(q-1,4)-15/4\cdot(q,5)+23462\cdot(q-1,5)-61/2\cdot(q-2,5)-61/2\cdot(q-3,5)\\
&\qquad-1/3\cdot(q,7)+118/3\cdot(q-1,7)-1/6\cdot(q-2,7)-1/2\cdot(q-3,7)-1/6\cdot(q-4,7)\\
&\qquad-1/2\cdot(q-5,7)+9\cdot(q-1,8)+2230441657/12]\cdot q^{16}\\
&+[-22986\cdot(q,2)\cdot(q-1,3)-53619055\cdot(q,2)+6135/2\cdot(q,3)+6334473\cdot(q-1,3)\\
&\qquad+2143215\cdot(q-1,4)+5\cdot(q,5)+55626\cdot(q-1,5)-25\cdot(q-2,5)-25\cdot(q-3,5)\\
&\qquad+688/3\cdot(q-1,7)-1/6\cdot(q-2,7)-1/2\cdot(q-3,7)-1/6\cdot(q-4,7)-1/2\cdot(q-5,7)\\
&\qquad+104\cdot(q-1,8)+396657133/2]\cdot q^{15}\\
&+[-63639\cdot(q,2)\cdot(q-1,3)-59237213\cdot(q,2)+2723\cdot(q,3)+8235551\cdot(q-1,3)\\
&\qquad+3140578\cdot(q-1,4)-75/2\cdot(q,5)+120323\cdot(q-1,5)-69\cdot(q-2,5)-69\cdot(q-3,5)\\
&\qquad-17/6\cdot(q,7)+998\cdot(q-1,7)-5/2\cdot(q-2,7)-7/2\cdot(q-3,7)-5/2\cdot(q-4,7)\\
&\qquad-7/2\cdot(q-5,7)+630\cdot(q-1,8)+628525168/3]\cdot q^{14}\\
&+[-156193\cdot(q,2)\cdot(q-1,3)-128984223/2\cdot(q,2)+2901\cdot(q,3)+10554561\cdot(q-1,3)\\
&\qquad+4477037\cdot(q-1,4)-85/4\cdot(q,5)+240018\cdot(q-1,5)-105/2\cdot(q-2,5)-105/2\cdot(q-3,5)\\
&\qquad-3\cdot(q,7)+10715/3\cdot(q-1,7)-11/6\cdot(q-2,7)-7/2\cdot(q-3,7)-11/6\cdot(q-4,7)\\
&\qquad-7/2\cdot(q-5,7)+10993/4\cdot(q-1,8)-1/2\cdot(q-3,8)+3/4\cdot(q-5,8)+874416491/4]\cdot q^{13}\\
&+[-345295\cdot(q,2)\cdot(q-1,3)-(q,2)\cdot(q-1,5)+1/2\cdot(q,2)\cdot(q-2,5)
\end{aligned}
$$

$$
\begin{aligned}
&+1/2 \cdot (q,2) \cdot (q-3,5) - 69026911 \cdot (q,2) + 4287/2 \cdot (q,3) + 13352309 \cdot (q-1,3)\\
&+6202586 \cdot (q-1,4) - 199/2 \cdot (q,5) + 444052 \cdot (q-1,5) - 129 \cdot (q-2,5) - 129 \cdot (q-3,5)\\
&-77/6 \cdot (q,7) + 32674/3 \cdot (q-1,7) - 32/3 \cdot (q-2,7) - 13 \cdot (q-3,7) - 32/3 \cdot (q-4,7)\\
&-13 \cdot (q-5,7) + 18971/2 \cdot (q-1,8) + 1/2 \cdot (q-3,8) - 3 \cdot (q-5,8) + 22/3 \cdot (q-1,9) - (q-2,9)\\
&-5/3 \cdot (q-4,9) - (q-5,9) - 5/3 \cdot (q-7,9) + 1348031285/6] \cdot q^{12}\\
+ [&-695814 \cdot (q,2) \cdot (q-1,3) - 46 \cdot (q,2) \cdot (q-1,5) + 1/2 \cdot (q,2) \cdot (q-2,5)\\
&+1/2 \cdot (q,2) \cdot (q-3,5) - 72398470 \cdot (q,2) + 1848 \cdot (q,3) + 16684881 \cdot (q-1,3)\\
&+8335252 \cdot (q-1,4) - 247/4 \cdot (q,5) + 765953 \cdot (q-1,5) - 187/2 \cdot (q-2,5)\\
&-187/2 \cdot (q-3,5) - 12 \cdot (q,7) + 87524/3 \cdot (q-1,7) - 25/3 \cdot (q-2,7) - 12 \cdot (q-3,7)\\
&-25/3 \cdot (q-4,7) - 12 \cdot (q-5,7) + 27631 \cdot (q-1,8) - (q-3,8) - 22 \cdot (q-5,8)\\
&+430/3 \cdot (q-1,9) - (q-2,9) - 5/3 \cdot (q-4,9) - (q-5,9) - 5/3 \cdot (q-7,9) + 906476203/4] \cdot q^{11}\\
+ [&-1287967 \cdot (q,2) \cdot (q-1,3) - 611 \cdot (q,2) \cdot (q-1,5) + 5/2 \cdot (q,2) \cdot (q-2,5)\\
&+5/2 \cdot (q,2) \cdot (q-3,5) + 14 \cdot (q-1,3) \cdot (q-1,4) - 592773599/8 \cdot (q,2) + 372 \cdot (q,3)\\
&+20569396 \cdot (q-1,3) + 43319501/4 \cdot (q-1,4) - 341/2 \cdot (q,5) + 1235882 \cdot (q-1,5)\\
&-419/2 \cdot (q-2,5) - 419/2 \cdot (q-3,5) - 67/2 \cdot (q,7) + 69721 \cdot (q-1,7) - 57/2 \cdot (q-2,7)\\
&-67/2 \cdot (q-3,7) - 57/2 \cdot (q-4,7) - 67/2 \cdot (q-5,7) + 1129047/16 \cdot (q-1,8)\\
&+19/2 \cdot (q-3,8) - 1329/16 \cdot (q-5,8) + 1159 \cdot (q-1,9) - 6 \cdot (q-2,9) - 10 \cdot (q-4,9)\\
&-6 \cdot (q-5,9) - 10 \cdot (q-7,9) + 10 \cdot (q-1,11) + 1786495589/8] \cdot q^{10}\\
+ [&-2201367 \cdot (q,2) \cdot (q-1,3) - 4475 \cdot (q,2) \cdot (q-1,5) + 2 \cdot (q,2) \cdot (q-2,5)\\
&+2 \cdot (q,2) \cdot (q-3,5) + 7 \cdot (q,3) \cdot (q-1,4) + 293 \cdot (q-1,3) \cdot (q-1,4) - 147149323/2 \cdot (q,2)\\
&-1181/2 \cdot (q,3) + 24938512 \cdot (q-1,3) + 13541168 \cdot (q-1,4) - 383/4 \cdot (q,5)\\
&+1873090 \cdot (q-1,5) - 139 \cdot (q-2,5) - 139 \cdot (q-3,5) - 55/2 \cdot (q,7) + 450868/3 \cdot (q-1,7)\\
&-127/6 \cdot (q-2,7) - 55/2 \cdot (q-3,7) - 127/6 \cdot (q-4,7) - 55/2 \cdot (q-5,7)\\
&+647007/4 \cdot (q-1,8) + 6 \cdot (q-3,8) - 869/4 \cdot (q-5,8) + 6139 \cdot (q-1,9) - 6 \cdot (q-2,9)\\
&-10 \cdot (q-4,9) - 6 \cdot (q-5,9) - 10 \cdot (q-7,9) + 156 \cdot (q-1,11) + 854664357/4] \cdot q^{9}\\
+ [&-3479212 \cdot (q,2) \cdot (q-1,3) - 21469 \cdot (q,2) \cdot (q-1,5) + 4 \cdot (q,2) \cdot (q-2,5)\\
&+4 \cdot (q,2) \cdot (q-3,5) - 88 \cdot (q,2) \cdot (q-1,7) + 69/2 \cdot (q,3) \cdot (q-1,4) - 1/4 \cdot (q,3) \cdot (q-2,5)\\
&-1/4 \cdot (q,3) \cdot (q-3,5) + 2667 \cdot (q-1,3) \cdot (q-1,4) - 3/4 \cdot (q-1,3) \cdot (q,5)\\
&+3 \cdot (q-1,3) \cdot (q-1,5) - (q-1,3) \cdot (q-2,5) - (q-1,3) \cdot (q-3,5) - 140662417/2 \cdot (q,2)\\
&-5291/2 \cdot (q,3) + 118200183/4 \cdot (q-1,3) + 32367431/2 \cdot (q-1,4) - 461/2 \cdot (q,5)\\
&+2676303 \cdot (q-1,5) - 1077/4 \cdot (q-2,5) - 1077/4 \cdot (q-3,5) - 123/2 \cdot (q,7)\\
&+878113/3 \cdot (q-1,7) - 319/6 \cdot (q-2,7) - 123/2 \cdot (q-3,7) - 319/6 \cdot (q-4,7)\\
&-123/2 \cdot (q-5,7) + 1343425/4 \cdot (q-1,8) + 49 \cdot (q-3,8) - 1671/4 \cdot (q-5,8)\\
&+70168/3 \cdot (q-1,9) - 16 \cdot (q-2,9) - 80/3 \cdot (q-4,9) - 16 \cdot (q-5,9) - 80/3 \cdot (q-7,9)\\
&+1144 \cdot (q-1,11) + 15 \cdot (q-1,13) + 196919313] \cdot q^{8}\\
+ [&-5064798 \cdot (q,2) \cdot (q-1,3) - 74074 \cdot (q,2) \cdot (q-1,5) + 2 \cdot (q,2) \cdot (q-2,5)\\
&+2 \cdot (q,2) \cdot (q-3,5) - 1162 \cdot (q,2) \cdot (q-1,7) + 215/2 \cdot (q,3) \cdot (q-1,4)\\
&+14464 \cdot (q-1,3) \cdot (q-1,4) - 2 \cdot (q-1,3) \cdot (q,5) + 53 \cdot (q-1,3) \cdot (q-1,5)\\
&-2 \cdot (q-1,3) \cdot (q-2,5) - 2 \cdot (q-1,3) \cdot (q-3,5) - 128077861/2 \cdot (q,2) - 4117 \cdot (q,3)\\
&+33860113 \cdot (q-1,3) + 36607605/2 \cdot (q-1,4) - 131 \cdot (q,5) + 3610280 \cdot (q-1,5)\\
&-156 \cdot (q-2,5) - 156 \cdot (q-3,5) - 43 \cdot (q,7) + 1539095/3 \cdot (q-1,7) - 94/3 \cdot (q-2,7)\\
&-43 \cdot (q-3,7) - 94/3 \cdot (q-4,7) - 43 \cdot (q-5,7) + 2513495/4 \cdot (q-1,8) + 34 \cdot (q-3,8)\\
&-2317/4 \cdot (q-5,8) + 203467/3 \cdot (q-1,9) - 29/2 \cdot (q-2,9) - 145/6 \cdot (q-4,9)\\
&-29/2 \cdot (q-5,9) - 145/6 \cdot (q-7,9) + 5227 \cdot (q-1,11) + 220 \cdot (q-1,13) + 105 \cdot (q-1,16)\\
&+345912571/2] \cdot q^{7}\\
+ [&-6704372 \cdot (q,2) \cdot (q-1,3) - 191354 \cdot (q,2) \cdot (q-1,5) + 3/2 \cdot (q,2) \cdot (q-2,5)\\
&+3/2 \cdot (q,2) \cdot (q-3,5) - 20180/3 \cdot (q,2) \cdot (q-1,7) - 2/3 \cdot (q,2) \cdot (q-2,7)\\
&-2/3 \cdot (q,2) \cdot (q-4,7) + 189 \cdot (q,3) \cdot (q-1,4) + 51904 \cdot (q-1,3) \cdot (q-1,4)\\
&-13/2 \cdot (q-1,3) \cdot (q,5) + 326 \cdot (q-1,3) \cdot (q-1,5) - 13/2 \cdot (q-1,3) \cdot (q-2,5)\\
&-13/2 \cdot (q-1,3) \cdot (q-3,5) - 109489081/2 \cdot (q,2) - 12595/2 \cdot (q,3)\\
&+73692335/2 \cdot (q-1,3) + 19289484 \cdot (q-1,4) - 1003/4 \cdot (q,5) + 4559641 \cdot (q-1,5)\\
&-259 \cdot (q-2,5) - 259 \cdot (q-3,5) - 80 \cdot (q,7) + 799328 \cdot (q-1,7) - 65 \cdot (q-2,7)
\end{aligned}
$$

$$
\begin{aligned}
&-80\cdot(q-3,7)-65\cdot(q-4,7)-80\cdot(q-5,7)+4161807/4\cdot(q-1,8)+128\cdot(q-3,8)\\
&-2213/4\cdot(q-5,8)+456595/3\cdot(q-1,9)-53/2\cdot(q-2,9)-265/6\cdot(q-4,9)\\
&-53/2\cdot(q-5,9)-265/6\cdot(q-7,9)+16346\cdot(q-1,11)+1330\cdot(q-1,13)+1050\cdot(q-1,16)\\
&+571264779/4]\cdot q^6\\
+&[-7864755\cdot(q,2)\cdot(q-1,3)-372780\cdot(q,2)\cdot(q-1,5)-(q,2)\cdot(q-2,5)\\
&-(q,2)\cdot(q-3,5)-22199\cdot(q,2)\cdot(q-1,7)+395/2\cdot(q,3)\cdot(q-1,4)\\
&+128533\cdot(q-1,3)\cdot(q-1,4)-17/2\cdot(q-1,3)\cdot(q,5)+1094\cdot(q-1,3)\cdot(q-1,5)\\
&-17/2\cdot(q-1,3)\cdot(q-2,5)-17/2\cdot(q-1,3)\cdot(q-3,5)-344733961/8\cdot(q,2)\\
&-15693/2\cdot(q,3)+73908015/2\cdot(q-1,3)+73944401/4\cdot(q-1,4)-459/4\cdot(q,5)\\
&+5247081\cdot(q-1,5)-225/2\cdot(q-2,5)-225/2\cdot(q-3,5)-42\cdot(q,7)\\
&+3236564/3\cdot(q-1,7)-97/3\cdot(q-2,7)-42\cdot(q-3,7)-97/3\cdot(q-4,7)-42\cdot(q-5,7)\\
&+23635401/16\cdot(q-1,8)+75\cdot(q-3,8)-2279/16\cdot(q-5,8)+788971/3\cdot(q-1,9)\\
&-35/2\cdot(q-2,9)-175/6\cdot(q-4,9)-35/2\cdot(q-5,9)-175/6\cdot(q-7,9)+35957\cdot(q-1,11)\\
&+4410\cdot(q-1,13)+4599\cdot(q-1,16)+872086317/8]\cdot q^5\\
+&[-7837102\cdot(q,2)\cdot(q-1,3)-538013\cdot(q,2)\cdot(q-1,5)-7/2\cdot(q,2)\cdot(q-2,5)\\
&-7/2\cdot(q,2)\cdot(q-3,5)-136432/3\cdot(q,2)\cdot(q-1,7)-1/3\cdot(q,2)\cdot(q-2,7)\\
&-1/3\cdot(q,2)\cdot(q-4,7)+94\cdot(q,3)\cdot(q-1,4)-(q,3)\cdot(q-2,5)-(q,3)\cdot(q-3,5)\\
&+220891\cdot(q-1,3)\cdot(q-1,4)-37/2\cdot(q-1,3)\cdot(q,5)+2189\cdot(q-1,3)\cdot(q-1,5)\\
&-39/2\cdot(q-1,3)\cdot(q-2,5)-39/2\cdot(q-1,3)\cdot(q-3,5)-243309807/8\cdot(q,2)\\
&-18155/2\cdot(q,3)+65217209/2\cdot(q-1,3)+62072253/4\cdot(q-1,4)-165\cdot(q,5)\\
&+5222047\cdot(q-1,5)-157\cdot(q-2,5)-157\cdot(q-3,5)-149/2\cdot(q,7)+1203844\cdot(q-1,7)\\
&-117/2\cdot(q-2,7)-149/2\cdot(q-3,7)-117/2\cdot(q-4,7)-149/2\cdot(q-5,7)\\
&+27482287/16\cdot(q-1,8)+381/2\cdot(q-3,8)+10711/16\cdot(q-5,8)+1023479/3\cdot(q-1,9)\\
&-61/2\cdot(q-2,9)-305/6\cdot(q-4,9)-61/2\cdot(q-5,9)-305/6\cdot(q-7,9)+55316\cdot(q-1,11)\\
&+8830\cdot(q-1,13)+11319\cdot(q-1,16)+602150549/8]\cdot q^4\\
+&[-6237443\cdot(q,2)\cdot(q-1,3)-552974\cdot(q,2)\cdot(q-1,5)-5/2\cdot(q,2)\cdot(q-2,5)\\
&-5/2\cdot(q,2)\cdot(q-3,5)-176861/3\cdot(q,2)\cdot(q-1,7)-5/3\cdot(q,2)\cdot(q-2,7)\\
&-5/3\cdot(q,2)\cdot(q-4,7)-133/2\cdot(q,3)\cdot(q-1,4)+257714\cdot(q-1,3)\cdot(q-1,4)\\
&-11\cdot(q-1,3)\cdot(q,5)+2729\cdot(q-1,3)\cdot(q-1,5)-11\cdot(q-1,3)\cdot(q-2,5)\\
&-11\cdot(q-1,3)\cdot(q-3,5)-37017477/2\cdot(q,2)-8396\cdot(q,3)+23687404\cdot(q-1,3)\\
&+21576261/2\cdot(q-1,4)-191/4\cdot(q,5)+4166274\cdot(q-1,5)-45\cdot(q-2,5)-45\cdot(q-3,5)\\
&-45/2\cdot(q,7)+1036943\cdot(q-1,7)-5/2\cdot(q-2,7)-45/2\cdot(q-3,7)-5/2\cdot(q-4,7)\\
&-45/2\cdot(q-5,7)+6150321/4\cdot(q-1,8)+80\cdot(q-3,8)+6901/4\cdot(q-5,8)+317967\cdot(q-1,9)\\
&-21/2\cdot(q-2,9)-35/2\cdot(q-4,9)-21/2\cdot(q-5,9)-35/2\cdot(q-7,9)+57829\cdot(q-1,11)\\
&+10960\cdot(q-1,13)+16737\cdot(q-1,16)+181838051/4]\cdot q^3\\
+&[-3632076\cdot(q,2)\cdot(q-1,3)-378687\cdot(q,2)\cdot(q-1,5)-4\cdot(q,2)\cdot(q-2,5)\\
&-4\cdot(q,2)\cdot(q-3,5)-47025\cdot(q,2)\cdot(q-1,7)-269/2\cdot(q,3)\cdot(q-1,4)\\
&+193452\cdot(q-1,3)\cdot(q-1,4)-41/2\cdot(q-1,3)\cdot(q,5)+2042\cdot(q-1,3)\cdot(q-1,5)\\
&-41/2\cdot(q-1,3)\cdot(q-2,5)-41/2\cdot(q-1,3)\cdot(q-3,5)-72743581/8\cdot(q,2)\\
&-13191/2\cdot(q,3)+25807691/2\cdot(q-1,3)+22834865/4\cdot(q-1,4)-261/4\cdot(q,5)\\
&+2410290\cdot(q-1,5)-61\cdot(q-2,5)-61\cdot(q-3,5)-45\cdot(q,7)+1884779/3\cdot(q-1,7)\\
&-121/3\cdot(q-2,7)-45\cdot(q-3,7)-121/3\cdot(q-4,7)-45\cdot(q-5,7)\\
&+15580069/16\cdot(q-1,8)+303/2\cdot(q-3,8)+43549/16\cdot(q-5,8)+595442/3\cdot(q-1,9)\\
&-43/2\cdot(q-2,9)-215/6\cdot(q-4,9)-43/2\cdot(q-5,9)-215/6\cdot(q-7,9)+38872\cdot(q-1,11)\\
&+8250\cdot(q-1,13)+14742\cdot(q-1,16)+180248845/8]\cdot q^2\\
+&[-1347376\cdot(q,2)\cdot(q-1,3)-153331\cdot(q,2)\cdot(q-1,5)-(q,2)\cdot(q-2,5)\\
&-(q,2)\cdot(q-3,5)-63100/3\cdot(q,2)\cdot(q-1,7)-1/3\cdot(q,2)\cdot(q-2,7)\\
&-1/3\cdot(q,2)\cdot(q-4,7)-193/2\cdot(q,3)\cdot(q-1,4)+83662\cdot(q-1,3)\cdot(q-1,4)\\
&-9/2\cdot(q-1,3)\cdot(q,5)+858\cdot(q-1,3)\cdot(q-1,5)-9/2\cdot(q-1,3)\cdot(q-2,5)\\
&-9/2\cdot(q-1,3)\cdot(q-3,5)-25461615/8\cdot(q,2)-6525/2\cdot(q,3)+9129483/2\cdot(q-1,3)\\
&+7986847/4\cdot(q-1,4)-8\cdot(q,5)+873892\cdot(q-1,5)-6\cdot(q-2,5)-6\cdot(q-3,5)-5\cdot(q,7)\\
&+699325/3\cdot(q-1,7)+7/3\cdot(q-2,7)-5\cdot(q-3,7)+7/3\cdot(q-4,7)-5\cdot(q-5,7)
\end{aligned}
$$

$$+6124391/16\cdot(q-1,8)+65/2\cdot(q-3,8)+54447/16\cdot(q-5,8)+220618/3\cdot(q-1,9)$$
$$-5/2\cdot(q-2,9)-25/6\cdot(q-4,9)-5/2\cdot(q-5,9)-25/6\cdot(q-7,9)+15071\cdot(q-1,11)$$
$$+3450\cdot(q-1,13)+7119\cdot(q-1,16)+64373201/8]\cdot q$$
$$+[-234918\cdot(q,2)\cdot(q-1,3)-27545\cdot(q,2)\cdot(q-1,5)-(q,2)\cdot(q-2,5)$$
$$-(q,2)\cdot(q-3,5)-4031\cdot(q,2)\cdot(q-1,7)-(q,2)\cdot(q-2,7)-(q,2)\cdot(q-4,7)$$
$$-34\cdot(q,3)\cdot(q-1,4)-3/4\cdot(q,3)\cdot(q-2,5)-3/4\cdot(q,3)\cdot(q-3,5)$$
$$+15738\cdot(q-1,3)\cdot(q-1,4)-31/4\cdot(q-1,3)\cdot(q,5)+146\cdot(q-1,3)\cdot(q-1,5)$$
$$-17/2\cdot(q-1,3)\cdot(q-2,5)-17/2\cdot(q-1,3)\cdot(q-3,5)-4755445/8\cdot(q,2)$$
$$-1360\cdot(q,3)+3081427/4\cdot(q-1,3)+1320715/4\cdot(q-1,4)-13\cdot(q,5)+146266\cdot(q-1,5)$$
$$-39/4\cdot(q-2,5)-39/4\cdot(q-3,5)-25/2\cdot(q,7)+39253\cdot(q-1,7)-5/2\cdot(q-2,7)$$
$$-25/2\cdot(q-3,7)-5/2\cdot(q-4,7)-25/2\cdot(q-5,7)+1146117/16\cdot(q-1,8)+49\cdot(q-3,8)$$
$$+60117/16\cdot(q-5,8)+36341/3\cdot(q-1,9)-13/2\cdot(q-2,9)-65/6\cdot(q-4,9)$$
$$-13/2\cdot(q-5,9)-65/6\cdot(q-7,9)+2552\cdot(q-1,11)+615\cdot(q-1,13)+1449\cdot(q-1,16)$$
$$+12334915/8]$$

A.5.9 Dimension $d = 14$

$$N_{14,5}(q) = q^{120}+q^{119}+3\cdot q^{118}+5\cdot q^{117}+10\cdot q^{116}+16\cdot q^{115}+28\cdot q^{114}+43\cdot q^{113}+70\cdot q^{112}$$
$$+105\cdot q^{111}+160\cdot q^{110}+233\cdot q^{109}+343\cdot q^{108}+486\cdot q^{107}+692\cdot q^{106}$$
$$+961\cdot q^{105}+1333\cdot q^{104}+1814\cdot q^{103}+2464\cdot q^{102}+3294\cdot q^{101}+4392\cdot q^{100}$$
$$+5782\cdot q^{99}+7582\cdot q^{98}+9841\cdot q^{97}+12723\cdot q^{96}+16299\cdot q^{95}$$
$$+20798\cdot q^{94}+26334\cdot q^{93}+33207\cdot q^{92}+41589\cdot q^{91}+51882\cdot q^{90}$$
$$+64324\cdot q^{89}+79448\cdot q^{88}+97586\cdot q^{87}+119418\cdot q^{86}+145405\cdot q^{85}$$
$$+176417\cdot q^{84}+213052\cdot q^{83}+256419\cdot q^{82}+307302\cdot q^{81}+367070\cdot q^{80}$$
$$+436733\cdot q^{79}+517985\cdot q^{78}+612084\cdot q^{77}+721106\cdot q^{76}+846611\cdot q^{75}$$
$$+991087\cdot q^{74}+1156457\cdot q^{73}+1345693\cdot q^{72}+1561089\cdot q^{71}$$
$$+1806174\cdot q^{70}+2083684\cdot q^{69}+2397724\cdot q^{68}+2751524\cdot q^{67}$$
$$+3149846\cdot q^{66}+3596424\cdot q^{65}+[-(q,2)+4096744]\cdot q^{64}$$
$$+[-3\cdot(q,2)+4655101]\cdot q^{63}+[-12\cdot(q,2)+5277729]\cdot q^{62}$$
$$+[-32\cdot(q,2)+5969551]\cdot q^{61}+[-82\cdot(q,2)+6737642]\cdot q^{60}$$
$$+[-183\cdot(q,2)+7587557]\cdot q^{59}+[-387\cdot(q,2)+8527323]\cdot q^{58}$$
$$+[-761\cdot(q,2)+9563263]\cdot q^{57}+[-1439\cdot(q,2)+10704445]\cdot q^{56}$$
$$+[-2589\cdot(q,2)+11958098]\cdot q^{55}+[-4512\cdot(q,2)+13334587]\cdot q^{54}$$
$$+[-7594\cdot(q,2)+14842255]\cdot q^{53}+[-12453\cdot(q,2)+16493099]\cdot q^{52}$$
$$+[-19878\cdot(q,2)+18296968]\cdot q^{51}+[-31060\cdot(q,2)+20267952]\cdot q^{50}$$
$$+[-47496\cdot(q,2)+22418002]\cdot q^{49}+[-71338\cdot(q,2)+24764072]\cdot q^{48}$$
$$+[-105255\cdot(q,2)+27320962]\cdot q^{47}+[-152932\cdot(q,2)+30109553]\cdot q^{46}$$
$$+[-218887\cdot(q,2)+33148746]\cdot q^{45}+[-309142\cdot(q,2)+36464660]\cdot q^{44}$$
$$+[-430970\cdot(q,2)+40081840]\cdot q^{43}+[-593804\cdot(q,2)+44033562]\cdot q^{42}$$
$$+[-808839\cdot(q,2)+48351863]\cdot q^{41}$$
$$+[-1090223\cdot(q,2)+1/2\cdot(q,3)+(q-1,3)+106158663/2]\cdot q^{40}$$
$$+[-1454476\cdot(q,2)+4\cdot(q,3)+10\cdot(q-1,3)+58257819]\cdot q^{39}$$
$$+[-1921956\cdot(q,2)+21/2\cdot(q,3)+41\cdot(q-1,3)+127882993/2]\cdot q^{38}$$
$$+[-2516019\cdot(q,2)+59/2\cdot(q,3)+145\cdot(q-1,3)+140368603/2]\cdot q^{37}$$
$$+[-3264809\cdot(q,2)+59\cdot(q,3)+418\cdot(q-1,3)+77054331]\cdot q^{36}$$
$$+[-4199974\cdot(q,2)+122\cdot(q,3)+1100\cdot(q-1,3)+84619286]\cdot q^{35}$$

$$
\begin{aligned}
&+[-5358829\cdot(q,2)+417/2\cdot(q,3)+2613\cdot(q-1,3)+3\cdot(q-1,4)+185925341/2]\cdot q^{34}\\
&+[-6782512\cdot(q,2)+359\cdot(q,3)+5810\cdot(q-1,3)+17\cdot(q-1,4)+102166773]\cdot q^{33}\\
&+[-8518428\cdot(q,2)+1083/2\cdot(q,3)+12094\cdot(q-1,3)+73\cdot(q-1,4)+224660229/2]\cdot q^{32}\\
&+[-10617724\cdot(q,2)+1647/2\cdot(q,3)+23945\cdot(q-1,3)+248\cdot(q-1,4)+247095777/2]\cdot q^{31}\\
&+[-13138040\cdot(q,2)+2253/2\cdot(q,3)+45175\cdot(q-1,3)+740\cdot(q-1,4)+271861301/2]\cdot q^{30}\\
&+[-16139980\cdot(q,2)+1553\cdot(q,3)+81891\cdot(q-1,3)+1968\cdot(q-1,4)+149580793]\cdot q^{29}\\
&+[-19690099\cdot(q,2)+1953\cdot(q,3)+142911\cdot(q-1,3)+4798\cdot(q-1,4)+164612674]\cdot q^{28}\\
&+[-23855996\cdot(q,2)+2487\cdot(q,3)+241281\cdot(q-1,3)+10861\cdot(q-1,4)+181123950]\cdot q^{27}\\
&+[-28709080\cdot(q,2)+2914\cdot(q,3)+394740\cdot(q-1,3)+23099\cdot(q-1,4)+199215754]\cdot q^{26}\\
&+[-34317538\cdot(q,2)+6965/2\cdot(q,3)+627722\cdot(q-1,3)+46521\cdot(q-1,4)+437914967/2]\cdot q^{25}\\
&+[-40748222\cdot(q,2)+3836\cdot(q,3)+971527\cdot(q-1,3)+89330\cdot(q-1,4)+3/4\cdot(q,5)+(q-1,5)\\
&\qquad+961622721/4]\cdot q^{24}\\
&+[-48056071\cdot(q,2)+4353\cdot(q,3)+1466439\cdot(q-1,3)+164393\cdot(q-1,4)+11/4\cdot(q,5)\\
&\qquad+16\cdot(q-1,5)+1054235113/4]\cdot q^{23}\\
&+[-56284418\cdot(q,2)+4537\cdot(q,3)+2160845\cdot(q-1,3)+291168\cdot(q-1,4)+11/2\cdot(q,5)\\
&\qquad+110\cdot(q-1,5)-3/2\cdot(q-2,5)-3/2\cdot(q-3,5)+576748445/2]\cdot q^{22}\\
&+[-11\cdot(q,2)\cdot(q-1,3)-65449590\cdot(q,2)+4930\cdot(q,3)+3112846\cdot(q-1,3)\\
&\qquad+498092\cdot(q-1,4)+23/2\cdot(q,5)+537\cdot(q-1,5)-3/2\cdot(q-2,5)-3/2\cdot(q-3,5)\\
&\qquad+629421297/2]\cdot q^{21}\\
&+[-116\cdot(q,2)\cdot(q-1,3)-75538734\cdot(q,2)+4879\cdot(q,3)+4387475\cdot(q-1,3)\\
&\qquad+825180\cdot(q-1,4)+13\cdot(q,5)+2059\cdot(q-1,5)-7\cdot(q-2,5)-7\cdot(q-3,5)+342341307]\cdot q^{20}\\
&+[-745\cdot(q,2)\cdot(q-1,3)-86488934\cdot(q,2)+5152\cdot(q,3)+6057548\cdot(q-1,3)\\
&\qquad+1326968\cdot(q-1,4)+41/2\cdot(q,5)+6690\cdot(q-1,5)-7\cdot(q-2,5)-7\cdot(q-3,5)+741767237/2]\cdot q^{19}\\
&+[-3478\cdot(q,2)\cdot(q-1,3)-98182590\cdot(q,2)+4916\cdot(q,3)+8199536\cdot(q-1,3)\\
&\qquad+2074879\cdot(q-1,4)+45/4\cdot(q,5)+18977\cdot(q-1,5)-47/2\cdot(q-2,5)-47/2\cdot(q-3,5)\\
&\qquad+6\cdot(q-1,7)+1599239595/4]\cdot q^{18}\\
&+[-13053\cdot(q,2)\cdot(q-1,3)-110420491\cdot(q,2)+10389/2\cdot(q,3)+10895865\cdot(q-1,3)\\
&\qquad+3159016\cdot(q-1,4)+37/2\cdot(q,5)+48264\cdot(q-1,5)-22\cdot(q-2,5)-22\cdot(q-3,5)\\
&\qquad+1/6\cdot(q,7)+64\cdot(q-1,7)+16\cdot(q-1,8)+2570184851/6]\cdot q^{17}\\
&+[-41224\cdot(q,2)\cdot(q-1,3)-122915643\cdot(q,2)+4882\cdot(q,3)+14232769\cdot(q-1,3)\\
&\qquad+4687451\cdot(q-1,4)-55/4\cdot(q,5)+111607\cdot(q-1,5)-58\cdot(q-2,5)-58\cdot(q-3,5)\\
&\qquad-1/3\cdot(q,7)+377\cdot(q-1,7)-(q-2,7)-(q-3,7)-(q-4,7)-(q-5,7)+174\cdot(q-1,8)\\
&\qquad+5466927241/12]\cdot q^{16}\\
&+[-113366\cdot(q,2)\cdot(q-1,3)-135261223\cdot(q,2)+5190\cdot(q,3)+18307529\cdot(q-1,3)\\
&\qquad+6782487\cdot(q-1,4)-11/2\cdot(q,5)+237530\cdot(q-1,5)-105/2\cdot(q-2,5)-105/2\cdot(q-3,5)\\
&\qquad+4937/3\cdot(q-1,7)-1/3\cdot(q-2,7)-(q-3,7)-1/3\cdot(q-4,7)-(q-5,7)+1027\cdot(q-1,8)\\
&\qquad-(q-3,8)+960359681/2]\cdot q^{15}\\
&+[-277448\cdot(q,2)\cdot(q-1,3)-146924797\cdot(q,2)+4581\cdot(q,3)+23228038\cdot(q-1,3)\\
&\qquad+9569858\cdot(q-1,4)-75\cdot(q,5)+468666\cdot(q-1,5)-245/2\cdot(q-2,5)-245/2\cdot(q-3,5)\\
&\qquad-37/6\cdot(q,7)+17539/3\cdot(q-1,7)-20/3\cdot(q-2,7)-7\cdot(q-3,7)-20/3\cdot(q-4,7)\\
&\qquad-7\cdot(q-5,7)+4416\cdot(q-1,8)-(q-3,8)+(q-5,8)+3003825589/6]\cdot q^{14}\\
&+[-614108\cdot(q,2)\cdot(q-1,3)-(q,2)\cdot(q-1,5)-314435389/2\cdot(q,2)+4481\cdot(q,3)\\
&\qquad+29116485\cdot(q-1,3)+13159240\cdot(q-1,4)-237/4\cdot(q,5)+862677\cdot(q-1,5)\\
&\qquad-106\cdot(q-2,5)-106\cdot(q-3,5)-13/2\cdot(q,7)+17794\cdot(q-1,7)-6\cdot(q-2,7)-7\cdot(q-3,7)\\
&\qquad-6\cdot(q-4,7)-7\cdot(q-5,7)+60683/4\cdot(q-1,8)-5\cdot(q-3,8)-25/4\cdot(q-5,8)+6\cdot(q-1,9)\\
&\qquad+2060360813/4]\cdot q^{13}\\
&+[-1242933\cdot(q,2)\cdot(q-1,3)-35\cdot(q,2)\cdot(q-1,5)+1/2\cdot(q,2)\cdot(q-2,5)\\
&\qquad+1/2\cdot(q,2)\cdot(q-3,5)-330607783/2\cdot(q,2)+3000\cdot(q,3)+36088321\cdot(q-1,3)\\
&\qquad+17609403\cdot(q-1,4)-171\cdot(q,5)+1487129\cdot(q-1,5)-217\cdot(q-2,5)-217\cdot(q-3,5)\\
&\qquad-76/3\cdot(q,7)+47617\cdot(q-1,7)-45/2\cdot(q-2,7)-51/2\cdot(q-3,7)-45/2\cdot(q-4,7)
\end{aligned}
$$

$$
\begin{aligned}
&-51/2\cdot(q-5,7)+176629/4\cdot(q-1,8)-5/2\cdot(q-3,8)-181/4\cdot(q-5,8)+114\cdot(q-1,9)\\
&-3/2\cdot(q-2,9)-5/2\cdot(q-4,9)-3/2\cdot(q-5,9)-5/2\cdot(q-7,9)+3128732909/6]\cdot q^{12}\\
+&[-2319209\cdot(q,2)\cdot(q-1,3)-513\cdot(q,2)\cdot(q-1,5)+1/2\cdot(q,2)\cdot(q-2,5)\\
&+1/2\cdot(q,2)\cdot(q-3,5)+2\cdot(q,3)\cdot(q-1,4)+12\cdot(q-1,3)\cdot(q-1,4)\\
&-1361580679/8\cdot(q,2)+1973\cdot(q,3)+44215911\cdot(q-1,3)+91510861/4\cdot(q-1,4)\\
&-517/4\cdot(q,5)+2409032\cdot(q-1,5)-363/2\cdot(q-2,5)-363/2\cdot(q-3,5)-49/2\cdot(q,7)\\
&+114262\cdot(q-1,7)-45/2\cdot(q-2,7)-49/2\cdot(q-3,7)-45/2\cdot(q-4,7)-49/2\cdot(q-5,7)\\
&+1805367/16\cdot(q-1,8)-17/2\cdot(q-3,8)-2737/16\cdot(q-5,8)+1067\cdot(q-1,9)\\
&-3/2\cdot(q-2,9)-5/2\cdot(q-4,9)-3/2\cdot(q-5,9)-5/2\cdot(q-7,9)+5\cdot(q-1,11)\\
&+4139706275/8]\cdot q^{11}\\
+&[-4011482\cdot(q,2)\cdot(q-1,3)-4130\cdot(q,2)\cdot(q-1,5)+9/2\cdot(q,2)\cdot(q-2,5)\\
&+9/2\cdot(q,2)\cdot(q-3,5)+7\cdot(q,3)\cdot(q-1,4)+228\cdot(q-1,3)\cdot(q-1,4)\\
&-683329361/4\cdot(q,2)-709\cdot(q,3)+53442639\cdot(q-1,3)+57506219/2\cdot(q-1,4)\\
&-549/2\cdot(q,5)+3677486\cdot(q-1,5)-675/2\cdot(q-2,5)-675/2\cdot(q-3,5)-62\cdot(q,7)\\
&+248576\cdot(q-1,7)-58\cdot(q-2,7)-62\cdot(q-3,7)-58\cdot(q-4,7)-62\cdot(q-5,7)\\
&+2075261/8\cdot(q-1,8)+9\cdot(q-3,8)-3627/8\cdot(q-5,8)+6261\cdot(q-1,9)-21/2\cdot(q-2,9)\\
&-35/2\cdot(q-4,9)-21/2\cdot(q-5,9)-35/2\cdot(q-7,9)+104\cdot(q-1,11)+2003723353/4]\cdot q^{10}\\
+&[-6452766\cdot(q,2)\cdot(q-1,3)-22045\cdot(q,2)\cdot(q-1,5)+2\cdot(q,2)\cdot(q-2,5)\\
&+2\cdot(q,2)\cdot(q-3,5)-43\cdot(q,2)\cdot(q-1,7)+41\cdot(q,3)\cdot(q-1,4)+2256\cdot(q-1,3)\cdot(q-1,4)\\
&-1/2\cdot(q-1,3)\cdot(q,5)+(q-1,3)\cdot(q-1,5)-1/2\cdot(q-1,3)\cdot(q-2,5)\\
&-1/2\cdot(q-1,3)\cdot(q-3,5)-166154533\cdot(q,2)-5459/2\cdot(q,3)+126927005/2\cdot(q-1,3)\\
&+34785007\cdot(q-1,4)-391/2\cdot(q,5)+5308623\cdot(q-1,5)-517/2\cdot(q-2,5)\\
&-517/2\cdot(q-3,5)-56\cdot(q,7)+1480099/3\cdot(q-1,7)-146/3\cdot(q-2,7)-56\cdot(q-3,7)\\
&-146/3\cdot(q-4,7)-56\cdot(q-5,7)+1088383/2\cdot(q-1,8)+13/2\cdot(q-3,8)-919\cdot(q-5,8)\\
&+26552\cdot(q-1,9)-21/2\cdot(q-2,9)-35/2\cdot(q-4,9)-21/2\cdot(q-5,9)-35/2\cdot(q-7,9)\\
&+941\cdot(q-1,11)+6\cdot(q-1,13)+470032186]\cdot q^{9}\\
+&[-9649682\cdot(q,2)\cdot(q-1,3)-85234\cdot(q,2)\cdot(q-1,5)+17/2\cdot(q,2)\cdot(q-2,5)\\
&+17/2\cdot(q,2)\cdot(q-3,5)-744\cdot(q,2)\cdot(q-1,7)+225/2\cdot(q,3)\cdot(q-1,4)\\
&-1/4\cdot(q,3)\cdot(q-2,5)-1/4\cdot(q,3)\cdot(q-3,5)+13674\cdot(q-1,3)\cdot(q-1,4)\\
&-5/4\cdot(q-1,3)\cdot(q,5)+31\cdot(q-1,3)\cdot(q-1,5)-3/2\cdot(q-1,3)\cdot(q-2,5)\\
&-3/2\cdot(q-1,3)\cdot(q-3,5)-155335743\cdot(q,2)-12501/2\cdot(q,3)\\
&+294098185/4\cdot(q-1,3)+80442981/2\cdot(q-1,4)-361\cdot(q,5)+7267536\cdot(q-1,5)\\
&-1669/4\cdot(q-2,5)-1669/4\cdot(q-3,5)-106\cdot(q,7)+2678714/3\cdot(q-1,7)\\
&-301/3\cdot(q-2,7)-106\cdot(q-3,7)-301/3\cdot(q-4,7)-106\cdot(q-5,7)+1044439\cdot(q-1,8)\\
&+135/2\cdot(q-3,8)-2911/2\cdot(q-5,8)+85966\cdot(q-1,9)-30\cdot(q-2,9)-50\cdot(q-4,9)\\
&-30\cdot(q-5,9)-50\cdot(q-7,9)+5152\cdot(q-1,11)+129\cdot(q-1,13)+63\cdot(q-1,16)\\
&+847855055/2]\cdot q^{8}\\
+&[-13336502\cdot(q,2)\cdot(q-1,3)-250614\cdot(q,2)\cdot(q-1,5)+2\cdot(q,2)\cdot(q-2,5)\\
&+2\cdot(q,2)\cdot(q-3,5)-5590\cdot(q,2)\cdot(q-1,7)+493/2\cdot(q,3)\cdot(q-1,4)\\
&+56843\cdot(q-1,3)\cdot(q-1,4)-6\cdot(q-1,3)\cdot(q,5)+263\cdot(q-1,3)\cdot(q-1,5)\\
&-6\cdot(q-1,3)\cdot(q-2,5)-6\cdot(q-1,3)\cdot(q-3,5)-552252437/4\cdot(q,2)-8808\cdot(q,3)\\
&+82149148\cdot(q-1,3)+43988304\cdot(q-1,4)-1005/4\cdot(q,5)+9427875\cdot(q-1,5)\\
&-565/2\cdot(q-2,5)-565/2\cdot(q-3,5)-175/2\cdot(q,7)+1464623\cdot(q-1,7)\\
&-157/2\cdot(q-2,7)-175/2\cdot(q-3,7)-157/2\cdot(q-4,7)-175/2\cdot(q-5,7)\\
&+14540441/8\cdot(q-1,8)+63\cdot(q-3,8)-14047/8\cdot(q-5,8)+219163\cdot(q-1,9)\\
&-57/2\cdot(q-2,9)-95/2\cdot(q-4,9)-57/2\cdot(q-5,9)-95/2\cdot(q-7,9)+19331\cdot(q-1,11)\\
&+1076\cdot(q-1,13)+819\cdot(q-1,16)+726970099/2]\cdot q^{7}\\
+&[-16784647\cdot(q,2)\cdot(q-1,3)-570797\cdot(q,2)\cdot(q-1,5)+9/2\cdot(q,2)\cdot(q-2,5)\\
&+9/2\cdot(q,2)\cdot(q-3,5)-72470/3\cdot(q,2)\cdot(q-1,7)-2/3\cdot(q,2)\cdot(q-2,7)\\
&-2/3\cdot(q,2)\cdot(q-4,7)+687/2\cdot(q,3)\cdot(q-1,4)+168997\cdot(q-1,3)\cdot(q-1,4)\\
&-21/2\cdot(q-1,3)\cdot(q,5)+1208\cdot(q-1,3)\cdot(q-1,5)-21/2\cdot(q-1,3)\cdot(q-2,5)\\
&-21/2\cdot(q-1,3)\cdot(q-3,5)-459827357/4\cdot(q,2)-12299\cdot(q,3)
\end{aligned}
$$

$$
\begin{aligned}
&\quad +173678439/2 \cdot (q-1,3) + 44775341 \cdot (q-1,4) - 1437/4 \cdot (q,5) + 11454274 \cdot (q-1,5) \\
&\quad -743/2 \cdot (q-2,5) - 743/2 \cdot (q-3,5) - 251/2 \cdot (q,7) + 2146101 \cdot (q-1,7) \\
&\quad -217/2 \cdot (q-2,7) - 251/2 \cdot (q-3,7) - 217/2 \cdot (q-4,7) - 251/2 \cdot (q-5,7) \\
&\quad +22477333/8 \cdot (q-1,8) + 188 \cdot (q-3,8) - 11443/8 \cdot (q-5,8) + 442653 \cdot (q-1,9) \\
&\quad -93/2 \cdot (q-2,9) - 155/2 \cdot (q-4,9) - 93/2 \cdot (q-5,9) - 155/2 \cdot (q-7,9) + 52210 \cdot (q-1,11) \\
&\quad +4874 \cdot (q-1,13) + 4704 \cdot (q-1,16) + 584455365/2] \cdot q^6 \\
&+ [-18722355 \cdot (q,2) \cdot (q-1,3) - 1003272 \cdot (q,2) \cdot (q-1,5) - (q,2) \cdot (q-2,5) \\
&\quad -(q,2) \cdot (q-3,5) - 66262 \cdot (q,2) \cdot (q-1,7) + 292 \cdot (q,3) \cdot (q-1,4) \\
&\quad +363842 \cdot (q-1,3) \cdot (q-1,4) - 37/2 \cdot (q-1,3) \cdot (q,5) + 3358 \cdot (q-1,3) \cdot (q-1,5) \\
&\quad -37/2 \cdot (q-1,3) \cdot (q-2,5) - 37/2 \cdot (q-1,3) \cdot (q-3,5) - 703433655/8 \cdot (q,2) \\
&\quad -14342 \cdot (q,3) + 168454283/2 \cdot (q-1,3) + 165504265/4 \cdot (q-1,4) - 839/4 \cdot (q,5) \\
&\quad +12645863 \cdot (q-1,5) - 207 \cdot (q-2,5) - 207 \cdot (q-3,5) - 175/2 \cdot (q,7) \\
&\quad +8194382/3 \cdot (q-1,7) - 479/6 \cdot (q-2,7) - 175/2 \cdot (q-3,7) - 479/6 \cdot (q-4,7) \\
&\quad -175/2 \cdot (q-5,7) + 59757831/16 \cdot (q-1,8) + 279/2 \cdot (q-3,8) - 4161/16 \cdot (q-5,8) \\
&\quad +700449 \cdot (q-1,9) - 75/2 \cdot (q-2,9) - 125/2 \cdot (q-4,9) - 75/2 \cdot (q-5,9) - 125/2 \cdot (q-7,9) \\
&\quad +102529 \cdot (q-1,11) + 13506 \cdot (q-1,13) + 15561 \cdot (q-1,16) + 1732982199/8] \cdot q^5 \\
&+ [-17741003 \cdot (q,2) \cdot (q-1,3) - 1329892 \cdot (q,2) \cdot (q-1,5) - 13/2 \cdot (q,2) \cdot (q-2,5) \\
&\quad -13/2 \cdot (q,2) \cdot (q-3,5) - 119450 \cdot (q,2) \cdot (q-1,7) + 183/2 \cdot (q,3) \cdot (q-1,4) \\
&\quad -(q,3) \cdot (q-2,5) - (q,3) \cdot (q-3,5) + 561696 \cdot (q-1,3) \cdot (q-1,4) \\
&\quad -57/2 \cdot (q-1,3) \cdot (q,5) + 5957 \cdot (q-1,3) \cdot (q-1,5) - 59/2 \cdot (q-1,3) \cdot (q-2,5) \\
&\quad -59/2 \cdot (q-1,3) \cdot (q-3,5) - 481352841/8 \cdot (q,2) - 31161/2 \cdot (q,3) \\
&\quad +143166189/2 \cdot (q-1,3) + 133713337/4 \cdot (q-1,4) - 397/2 \cdot (q,5) + 12039006 \cdot (q-1,5) \\
&\quad -188 \cdot (q-2,5) - 188 \cdot (q-3,5) - 101 \cdot (q,7) + 2881110 \cdot (q-1,7) - 90 \cdot (q-2,7) \\
&\quad -101 \cdot (q-3,7) - 90 \cdot (q-4,7) - 101 \cdot (q-5,7) + 65278273/16 \cdot (q-1,8) + 569/2 \cdot (q-3,8) \\
&\quad +28345/16 \cdot (q-5,8) + 2533019/3 \cdot (q-1,9) - 44 \cdot (q-2,9) - 220/3 \cdot (q-4,9) \\
&\quad -44 \cdot (q-5,9) - 220/3 \cdot (q-7,9) + 144348 \cdot (q-1,11) + 23942 \cdot (q-1,13) + 32193 \cdot (q-1,16) \\
&\quad +1159711075/8] \cdot q^4 \\
&+ [-13443589 \cdot (q,2) \cdot (q-1,3) - 1274558 \cdot (q,2) \cdot (q-1,5) - 5/2 \cdot (q,2) \cdot (q-2,5) \\
&\quad -5/2 \cdot (q,2) \cdot (q-3,5) - 423320/3 \cdot (q,2) \cdot (q-1,7) - 5/3 \cdot (q,2) \cdot (q-2,7) \\
&\quad -5/3 \cdot (q,2) \cdot (q-4,7) - 155 \cdot (q,3) \cdot (q-1,4) + 602959 \cdot (q-1,3) \cdot (q-1,4) \\
&\quad -21 \cdot (q-1,3) \cdot (q,5) + 6791 \cdot (q-1,3) \cdot (q-1,5) - 21 \cdot (q-1,3) \cdot (q-2,5) \\
&\quad -21 \cdot (q-1,3) \cdot (q-3,5) - 283424975/8 \cdot (q,2) - 27079/2 \cdot (q,3) + 49962930 \cdot (q-1,3) \\
&\quad +89347509/4 \cdot (q-1,4) - 91 \cdot (q,5) + 9183109 \cdot (q-1,5) - 88 \cdot (q-2,5) - 88 \cdot (q-3,5) \\
&\quad -49 \cdot (q,7) + 7065350/3 \cdot (q-1,7) - 79/3 \cdot (q-2,7) - 49 \cdot (q-3,7) - 79/3 \cdot (q-4,7) \\
&\quad -49 \cdot (q-5,7) + 55164471/16 \cdot (q-1,8) + 291/2 \cdot (q-3,8) + 64831/16 \cdot (q-5,8) \\
&\quad +740664 \cdot (q-1,9) - 24 \cdot (q-2,9) - 40 \cdot (q-4,9) - 24 \cdot (q-5,9) - 40 \cdot (q-7,9) \\
&\quad +140840 \cdot (q-1,11) + 27248 \cdot (q-1,13) + 42315 \cdot (q-1,16) + 678034921/8] \cdot q^3 \\
&+ [-7475083 \cdot (q,2) \cdot (q-1,3) - 824784 \cdot (q,2) \cdot (q-1,5) - 9 \cdot (q,2) \cdot (q-2,5) \\
&\quad -9 \cdot (q,2) \cdot (q-3,5) - 104996 \cdot (q,2) \cdot (q-1,7) - 229 \cdot (q,3) \cdot (q-1,4) \\
&\quad +424249 \cdot (q-1,3) \cdot (q-1,4) - 61/2 \cdot (q-1,3) \cdot (q,5) + 4780 \cdot (q-1,3) \cdot (q-1,5) \\
&\quad -61/2 \cdot (q-1,3) \cdot (q-2,5) - 61/2 \cdot (q-1,3) \cdot (q-3,5) - 33621987/2 \cdot (q,2) \\
&\quad -20201/2 \cdot (q,3) + 52302415/2 \cdot (q-1,3) + 11357448 \cdot (q-1,4) - 105/2 \cdot (q,5) \\
&\quad +5093403 \cdot (q-1,5) - 43 \cdot (q-2,5) - 43 \cdot (q-3,5) - 103/2 \cdot (q,7) + 4087048/3 \cdot (q-1,7) \\
&\quad -283/6 \cdot (q-2,7) - 103/2 \cdot (q-3,7) - 283/6 \cdot (q-4,7) - 103/2 \cdot (q-5,7) \\
&\quad +8297773/4 \cdot (q-1,8) + 227 \cdot (q-3,8) + 24501/4 \cdot (q-5,8) + 1320880/3 \cdot (q-1,9) \\
&\quad -25 \cdot (q-2,9) - 125/3 \cdot (q-4,9) - 25 \cdot (q-5,9) - 125/3 \cdot (q-7,9) + 89738 \cdot (q-1,11) \\
&\quad +19242 \cdot (q-1,13) + 34272 \cdot (q-1,16) + 40649152] \cdot q^2 \\
&+ [-2658837 \cdot (q,2) \cdot (q-1,3) - 319201 \cdot (q,2) \cdot (q-1,5) - (q,2) \cdot (q-2,5) \\
&\quad -(q,2) \cdot (q-3,5) - 44537 \cdot (q,2) \cdot (q-1,7) - 309/2 \cdot (q,3) \cdot (q-1,4) \\
&\quad +174526 \cdot (q-1,3) \cdot (q-1,4) - 8 \cdot (q-1,3) \cdot (q,5) + 1909 \cdot (q-1,3) \cdot (q-1,5) \\
&\quad -8 \cdot (q-1,3) \cdot (q-2,5) - 8 \cdot (q-1,3) \cdot (q-3,5) - 45405341/8 \cdot (q,2) - 4824 \cdot (q,3) \\
&\quad +8906067 \cdot (q-1,3) + 15276461/4 \cdot (q-1,4) - 67/4 \cdot (q,5) + 1779092 \cdot (q-1,5)
\end{aligned}
$$

$$-29/2 \cdot (q-2,5) - 29/2 \cdot (q-3,5) - 23/2 \cdot (q,7) + 1457719/3 \cdot (q-1,7) - 43/6 \cdot (q-2,7)$$
$$-23/2 \cdot (q-3,7) - 43/6 \cdot (q-4,7) - 23/2 \cdot (q-5,7) + 12486069/16 \cdot (q-1,8) + 58 \cdot (q-3,8)$$
$$+118709/16 \cdot (q-5,8) + 156895 \cdot (q-1,9) - 6 \cdot (q-2,9) - 10 \cdot (q-4,9) - 6 \cdot (q-5,9)$$
$$-10 \cdot (q-7,9) + 33378 \cdot (q-1,11) + 7668 \cdot (q-1,13) + 15561 \cdot (q-1,16) + 112406729/8] \cdot q$$
$$+ [-446644 \cdot (q,2) \cdot (q-1,3) - 55332 \cdot (q,2) \cdot (q-1,5) - 5/2 \cdot (q,2) \cdot (q-2,5)$$
$$-5/2 \cdot (q,2) \cdot (q-3,5) - 8192 \cdot (q,2) \cdot (q-1,7) - (q,2) \cdot (q-2,7) - (q,2) \cdot (q-4,7)$$
$$-107/2 \cdot (q,3) \cdot (q-1,4) - 3/4 \cdot (q,3) \cdot (q-2,5) - 3/4 \cdot (q,3) \cdot (q-3,5)$$
$$+31594 \cdot (q-1,3) \cdot (q-1,4) - 45/4 \cdot (q-1,3) \cdot (q,5) + 318 \cdot (q-1,3) \cdot (q-1,5)$$
$$-12 \cdot (q-1,3) \cdot (q-2,5) - 12 \cdot (q-1,3) \cdot (q-3,5) - 8179443/8 \cdot (q,2) - 3871/2 \cdot (q,3)$$
$$+5806245/4 \cdot (q-1,3) + 2419091/4 \cdot (q-1,4) - 17/4 \cdot (q,5) + 288478 \cdot (q-1,5)$$
$$-1/4 \cdot (q-2,5) - 1/4 \cdot (q-3,5) - 25/2 \cdot (q,7) + 237550/3 \cdot (q-1,7) - 7/6 \cdot (q-2,7)$$
$$-25/2 \cdot (q-3,7) - 7/6 \cdot (q-4,7) - 25/2 \cdot (q-5,7) + 2260587/16 \cdot (q-1,8)$$
$$+147/2 \cdot (q-3,8) + 129235/16 \cdot (q-5,8) + 75227/3 \cdot (q-1,9) - 13/2 \cdot (q-2,9)$$
$$-65/6 \cdot (q-4,9) - 13/2 \cdot (q-5,9) - 65/6 \cdot (q-7,9) + 5472 \cdot (q-1,11) + 1317 \cdot (q-1,13)$$
$$+3024 \cdot (q-1,16) + 20869011/8]$$

A.5.10 Dimension $d = 15$

$$N_{15,5}(q) = q^{126} + q^{125} + 3 \cdot q^{124} + 5 \cdot q^{123} + 10 \cdot q^{122} + 16 \cdot q^{121} + 28 \cdot q^{120} + 43 \cdot q^{119} + 70 \cdot q^{118}$$
$$+ 105 \cdot q^{117} + 161 \cdot q^{116} + 234 \cdot q^{115} + 346 \cdot q^{114} + 491 \cdot q^{113} + 702 \cdot q^{112}$$
$$+ 977 \cdot q^{111} + 1360 \cdot q^{110} + 1856 \cdot q^{109} + 2531 \cdot q^{108} + 3394 \cdot q^{107} + 4543 \cdot q^{106}$$
$$+ 6000 \cdot q^{105} + 7900 \cdot q^{104} + 10289 \cdot q^{103} + 13355 \cdot q^{102} + 17171 \cdot q^{101}$$
$$+ 21998 \cdot q^{100} + 27957 \cdot q^{99} + 35395 \cdot q^{98} + 44497 \cdot q^{97} + 55733 \cdot q^{96}$$
$$+ 69363 \cdot q^{95} + 86015 \cdot q^{94} + 106061 \cdot q^{93} + 130309 \cdot q^{92} + 159282 \cdot q^{91}$$
$$+ 194023 \cdot q^{90} + 235227 \cdot q^{89} + 284232 \cdot q^{88} + 341958 \cdot q^{87} + 410080 \cdot q^{86}$$
$$+ 489797 \cdot q^{85} + 583202 \cdot q^{84} + 691811 \cdot q^{83} + 818208 \cdot q^{82} + 964304 \cdot q^{81}$$
$$+ 1133228 \cdot q^{80} + 1327360 \cdot q^{79} + 1550477 \cdot q^{78} + 1805461 \cdot q^{77}$$
$$+ 2096836 \cdot q^{76} + 2428076 \cdot q^{75} + 2804509 \cdot q^{74} + 3230283 \cdot q^{73}$$
$$+ 3711634 \cdot q^{72} + 4253422 \cdot q^{71} + 4862888 \cdot q^{70} + 5545692 \cdot q^{69}$$
$$+ 6310130 \cdot q^{68} + 7162729 \cdot q^{67} + [-2 \cdot (q,2) + 8112954] \cdot q^{66}$$
$$+ [-7 \cdot (q,2) + 9168212] \cdot q^{65} + [-24 \cdot (q,2) + 10339239] \cdot q^{64}$$
$$+ [-63 \cdot (q,2) + 11634428] \cdot q^{63} + [-156 \cdot (q,2) + 13065832] \cdot q^{62}$$
$$+ [-342 \cdot (q,2) + 14642936] \cdot q^{61} + [-712 \cdot (q,2) + 16379288] \cdot q^{60}$$
$$+ [-1385 \cdot (q,2) + 18285557] \cdot q^{59} + [-2583 \cdot (q,2) + 20377013] \cdot q^{58}$$
$$+ [-4603 \cdot (q,2) + 22665770] \cdot q^{57} + [-7940 \cdot (q,2) + 25169100] \cdot q^{56}$$
$$+ [-13245 \cdot (q,2) + 27900948] \cdot q^{55} + [-21530 \cdot (q,2) + 30881147] \cdot q^{54}$$
$$+ [-34101 \cdot (q,2) + 34126001] \cdot q^{53} + [-52875 \cdot (q,2) + 37658670] \cdot q^{52}$$
$$+ [-80299 \cdot (q,2) + 41498782] \cdot q^{51} + [-119796 \cdot (q,2) + 45673865] \cdot q^{50}$$
$$+ [-175662 \cdot (q,2) + 50208162] \cdot q^{49} + [-253714 \cdot (q,2) + 55135228] \cdot q^{48}$$
$$+ [-361119 \cdot (q,2) + 60485600] \cdot q^{47} + [-507304 \cdot (q,2) + 66301010] \cdot q^{46}$$
$$+ [-703705 \cdot (q,2) + 72620755] \cdot q^{45} + [-964959 \cdot (q,2) + 79497340] \cdot q^{44}$$
$$+ [-1308535 \cdot (q,2) + 86981796] \cdot q^{43}$$
$$+ [-1756294 \cdot (q,2) + 1/2 \cdot (q,3) + (q-1,3) + 190281643/2] \cdot q^{42}$$
$$+ [-2333838 \cdot (q,2) + 4 \cdot (q,3) + 10 \cdot (q-1,3) + 104040443] \cdot q^{41}$$
$$+ [-3072532 \cdot (q,2) + 21/2 \cdot (q,3) + 41 \cdot (q-1,3) + 227530369/2] \cdot q^{40}$$
$$+ [-4008482 \cdot (q,2) + 30 \cdot (q,3) + 146 \cdot (q-1,3) + 124399815] \cdot q^{39}$$

$$
\begin{aligned}
&+[-5184955\cdot(q,2)+61\cdot(q,3)+425\cdot(q-1,3)+136050045]\cdot q^{38}\\
&+[-6650869\cdot(q,2)+259/2\cdot(q,3)+1132\cdot(q-1,3)+297644943/2]\cdot q^{37}\\
&+[-8463678\cdot(q,2)+449/2\cdot(q,3)+2719\cdot(q-1,3)+2\cdot(q-1,4)+325693667/2]\cdot q^{36}\\
&+[-10687045\cdot(q,2)+793/2\cdot(q,3)+6122\cdot(q-1,3)+12\cdot(q-1,4)+356505941/2]\cdot q^{35}\\
&+[-13394263\cdot(q,2)+1219/2\cdot(q,3)+12898\cdot(q-1,3)+53\cdot(q-1,4)+390390389/2]\cdot q^{34}\\
&+[-16664919\cdot(q,2)+948\cdot(q,3)+25855\cdot(q-1,3)+185\cdot(q-1,4)+213825529]\cdot q^{33}\\
&+[-20588668\cdot(q,2)+2647/2\cdot(q,3)+49375\cdot(q-1,3)+569\cdot(q-1,4)+468638389/2]\cdot q^{32}\\
&+[-25260728\cdot(q,2)+1868\cdot(q,3)+90603\cdot(q-1,3)+1561\cdot(q-1,4)+256844134]\cdot q^{31}\\
&+[-30785968\cdot(q,2)+4799/2\cdot(q,3)+160036\cdot(q-1,3)+3930\cdot(q-1,4)+563174223/2]\cdot q^{30}\\
&+[-37272594\cdot(q,2)+3127\cdot(q,3)+273492\cdot(q-1,3)+9180\cdot(q-1,4)+308717286]\cdot q^{29}\\
&+[-44836296\cdot(q,2)+7483/2\cdot(q,3)+452882\cdot(q-1,3)+20115\cdot(q-1,4)+676825663/2]\cdot q^{28}\\
&+[-53591443\cdot(q,2)+4569\cdot(q,3)+729046\cdot(q-1,3)+41681\cdot(q-1,4)+370816584]\cdot q^{27}\\
&+[-63654095\cdot(q,2)+5140\cdot(q,3)+1142344\cdot(q-1,3)+82194\cdot(q-1,4)+1/4\cdot(q,5)\\
&\quad+1624240323/4]\cdot q^{26}\\
&+[-75129507\cdot(q,2)+11915/2\cdot(q,3)+1746054\cdot(q-1,3)+155107\cdot(q-1,4)+(q,5)\\
&\quad+2\cdot(q-1,5)+888418601/2]\cdot q^{25}\\
&+[-88112632\cdot(q,2)+6345\cdot(q,3)+2605933\cdot(q-1,3)+281270\cdot(q-1,4)+3\cdot(q,5)\\
&\quad+24\cdot(q-1,5)-(q-2,5)-(q-3,5)+485283388]\cdot q^{24}\\
&+[-102669892\cdot(q,2)+7042\cdot(q,3)+3803408\cdot(q-1,3)+492013\cdot(q-1,4)+33/4\cdot(q,5)\\
&\quad+159\cdot(q-1,5)-(q-2,5)-(q-3,5)+2116743275/4]\cdot q^{23}\\
&+[-10\cdot(q,2)\cdot(q-1,3)-118836001\cdot(q,2)+7123\cdot(q,3)+5432749\cdot(q-1,3)\\
&\quad+832604\cdot(q-1,4)+12\cdot(q,5)+743\cdot(q-1,5)-5\cdot(q-2,5)-5\cdot(q-3,5)+575716985]\cdot q^{22}\\
&+[-120\cdot(q,2)\cdot(q-1,3)-136588661\cdot(q,2)+7634\cdot(q,3)+7603112\cdot(q-1,3)\\
&\quad+1366669\cdot(q-1,4)+43/2\cdot(q,5)+2822\cdot(q-1,5)-5\cdot(q-2,5)-5\cdot(q-3,5)\\
&\quad+1248976759/2]\cdot q^{21}\\
&+[-797\cdot(q,2)\cdot(q-1,3)-155840506\cdot(q,2)+7371\cdot(q,3)+10432374\cdot(q-1,3)\\
&\quad+2180205\cdot(q-1,4)+18\cdot(q,5)+9078\cdot(q-1,5)-18\cdot(q-2,5)-18\cdot(q-3,5)+674926892]\cdot q^{20}\\
&+[-3855\cdot(q,2)\cdot(q-1,3)-176406647\cdot(q,2)+7769\cdot(q,3)+14048748\cdot(q-1,3)\\
&\quad+3386172\cdot(q-1,4)+57/2\cdot(q,5)+25743\cdot(q-1,5)-33/2\cdot(q-2,5)-33/2\cdot(q-3,5)\\
&\quad+4\cdot(q-1,7)+1452347741/2]\cdot q^{19}\\
&+[-14856\cdot(q,2)\cdot(q-1,3)-197992133\cdot(q,2)+7295\cdot(q,3)+18583783\cdot(q-1,3)\\
&\quad+5126828\cdot(q-1,4)+17/4\cdot(q,5)+65536\cdot(q-1,5)-48\cdot(q-2,5)-48\cdot(q-3,5)\\
&\quad-1/3\cdot(q,7)+154/3\cdot(q-1,7)-1/6\cdot(q-2,7)-1/2\cdot(q-3,7)-1/6\cdot(q-4,7)\\
&\quad-1/2\cdot(q-5,7)+14\cdot(q-1,8)+9325016065/12]\cdot q^{18}\\
&+[-48206\cdot(q,2)\cdot(q-1,3)-220152304\cdot(q,2)+15551/2\cdot(q,3)+24178711\cdot(q-1,3)\\
&\quad+7574806\cdot(q-1,4)+67/4\cdot(q,5)+152348\cdot(q-1,5)-42\cdot(q-2,5)-42\cdot(q-3,5)\\
&\quad+1039/3\cdot(q-1,7)-1/6\cdot(q-2,7)-1/2\cdot(q-3,7)-1/6\cdot(q-4,7)-1/2\cdot(q-5,7)\\
&\quad+159\cdot(q-1,8)+3304462535/4]\cdot q^{17}\\
&+[-135934\cdot(q,2)\cdot(q-1,3)-242277676\cdot(q,2)+7180\cdot(q,3)+30982451\cdot(q-1,3)\\
&\quad+10927921\cdot(q-1,4)-183/4\cdot(q,5)+326533\cdot(q-1,5)-108\cdot(q-2,5)-108\cdot(q-3,5)\\
&\quad-17/6\cdot(q,7)+4892/3\cdot(q-1,7)-10/3\cdot(q-2,7)-4\cdot(q-3,7)-10/3\cdot(q-4,7)\\
&\quad-4\cdot(q-5,7)+993\cdot(q-1,8)+10456011895/12]\cdot q^{16}\\
&+[-340998\cdot(q,2)\cdot(q-1,3)-527099401/2\cdot(q,2)+7595\cdot(q,3)+39165965\cdot(q-1,3)\\
&\quad+15397972\cdot(q-1,4)-51/2\cdot(q,5)+650779\cdot(q-1,5)-179/2\cdot(q-2,5)\\
&\quad-179/2\cdot(q-3,5)-17/6\cdot(q,7)+6124\cdot(q-1,7)-2\cdot(q-2,7)-4\cdot(q-3,7)-2\cdot(q-4,7)\\
&\quad-4\cdot(q-5,7)+17939/4\cdot(q-1,8)-(q-3,8)+3/4\cdot(q-5,8)+5461882547/6]\cdot q^{15}\\
&+[-773045\cdot(q,2)\cdot(q-1,3)-282928852\cdot(q,2)+6374\cdot(q,3)+48918071\cdot(q-1,3)\\
&\quad+21184601\cdot(q-1,4)-579/4\cdot(q,5)+1212365\cdot(q-1,5)-207\cdot(q-2,5)-207\cdot(q-3,5)\\
&\quad-97/6\cdot(q,7)+19385\cdot(q-1,7)-15\cdot(q-2,7)-17\cdot(q-3,7)-15\cdot(q-4,7)-17\cdot(q-5,7)\\
&\quad+16076\cdot(q-1,8)-1/2\cdot(q-3,8)-11/2\cdot(q-5,8)+2/3\cdot(q-1,9)-1/2\cdot(q-2,9)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -5/6\cdot(q-4,9)-1/2\cdot(q-5,9)-5/6\cdot(q-7,9)+11282257235/12]\cdot q^{14}\\
&+[-1602797\cdot(q,2)\cdot(q-1,3)-14\cdot(q,2)\cdot(q-1,5)-299122277\cdot(q,2)+5976\cdot(q,3)\\
&\quad +60442956\cdot(q-1,3)+28433083\cdot(q-1,4)-401/4\cdot(q,5)+2120718\cdot(q-1,5)\\
&\quad -327/2\cdot(q-2,5)-327/2\cdot(q-3,5)-16\cdot(q,7)+161008/3\cdot(q-1,7)-79/6\cdot(q-2,7)\\
&\quad -33/2\cdot(q-3,7)-79/6\cdot(q-4,7)-33/2\cdot(q-5,7)+48501\cdot(q-1,8)-9/2\cdot(q-3,8)\\
&\quad -93/2\cdot(q-5,8)+140/3\cdot(q-1,9)-1/2\cdot(q-2,9)-5/6\cdot(q-4,9)-1/2\cdot(q-5,9)\\
&\quad -5/6\cdot(q-7,9)+3830462013/4]\cdot q^{13}\\
&+[-3064792\cdot(q,2)\cdot(q-1,3)-248\cdot(q,2)\cdot(q-1,5)+2\cdot(q,2)\cdot(q-2,5)\\
&\quad +2\cdot(q,2)\cdot(q-3,5)+4\cdot(q-1,3)\cdot(q-1,4)-2484849501/8\cdot(q,2)+3362\cdot(q,3)\\
&\quad +73905816\cdot(q-1,3)+148657991/4\cdot(q-1,4)-1125/4\cdot(q,5)+3493237\cdot(q-1,5)\\
&\quad -697/2\cdot(q-2,5)-697/2\cdot(q-3,5)-293/6\cdot(q,7)+132625\cdot(q-1,7)-43\cdot(q-2,7)\\
&\quad -49\cdot(q-3,7)-43\cdot(q-4,7)-49\cdot(q-5,7)+2043117/16\cdot(q-1,8)+5/2\cdot(q-3,8)\\
&\quad -3227/16\cdot(q-5,8)+1676/3\cdot(q-1,9)-5\cdot(q-2,9)-25/3\cdot(q-4,9)-5\cdot(q-5,9)\\
&\quad -25/3\cdot(q-7,9)+(q-1,11)+23015320031/24]\cdot q^{12}\\
&+[-5439079\cdot(q,2)\cdot(q-1,3)-2374\cdot(q,2)\cdot(q-1,5)+2\cdot(q,2)\cdot(q-2,5)\\
&\quad +2\cdot(q,2)\cdot(q-3,5)+4\cdot(q,3)\cdot(q-1,4)+104\cdot(q-1,3)\cdot(q-1,4)-315659424\cdot(q,2)\\
&\quad +1558\cdot(q,3)+89348530\cdot(q-1,3)+47183165\cdot(q-1,4)-373/2\cdot(q,5)\\
&\quad +5433255\cdot(q-1,5)-262\cdot(q-2,5)-262\cdot(q-3,5)-89/2\cdot(q,7)+891314/3\cdot(q-1,7)\\
&\quad -233/6\cdot(q-2,7)-89/2\cdot(q-3,7)-233/6\cdot(q-4,7)-89/2\cdot(q-5,7)\\
&\quad +602381/2\cdot(q-1,8)-9/2\cdot(q-3,8)-590\cdot(q-5,8)+11981/3\cdot(q-1,9)-5\cdot(q-2,9)\\
&\quad -25/3\cdot(q-4,9)-5\cdot(q-5,9)-25/3\cdot(q-7,9)+181/5\cdot(q-1,11)+1/5\cdot(q-3,11)\\
&\quad +1/5\cdot(q-4,11)+1/5\cdot(q-5,11)+1/5\cdot(q-9,11)+940505541]\cdot q^{11}\\
&+[-8996733\cdot(q,2)\cdot(q-1,3)-14702\cdot(q,2)\cdot(q-1,5)+7\cdot(q,2)\cdot(q-2,5)\\
&\quad +7\cdot(q,2)\cdot(q-3,5)-11\cdot(q,2)\cdot(q-1,7)+20\cdot(q,3)\cdot(q-1,4)-1/4\cdot(q,3)\cdot(q-2,5)\\
&\quad -1/4\cdot(q,3)\cdot(q-3,5)+1187\cdot(q-1,3)\cdot(q-1,4)-1/4\cdot(q-1,3)\cdot(q,5)\\
&\quad -1/2\cdot(q-1,3)\cdot(q-2,5)-1/2\cdot(q-1,3)\cdot(q-3,5)-2500019971/8\cdot(q,2)\\
&\quad -5317/2\cdot(q,3)+426082521/4\cdot(q-1,3)+231857145/4\cdot(q-1,4)-1675/4\cdot(q,5)\\
&\quad +8000027\cdot(q-1,5)-2013/4\cdot(q-2,5)-2013/4\cdot(q-3,5)-103\cdot(q,7)\\
&\quad +1825940/3\cdot(q-1,7)-283/3\cdot(q-2,7)-103\cdot(q-3,7)-283/3\cdot(q-4,7)\\
&\quad -103\cdot(q-5,7)+10367947/16\cdot(q-1,8)+65/2\cdot(q-3,8)-21149/16\cdot(q-5,8)\\
&\quad +58993/3\cdot(q-1,9)-41/2\cdot(q-2,9)-205/6\cdot(q-4,9)-41/2\cdot(q-5,9)\\
&\quad -205/6\cdot(q-7,9)+2254/5\cdot(q-1,11)-1/5\cdot(q-3,11)-1/5\cdot(q-4,11)-1/5\cdot(q-5,11)\\
&\quad -1/5\cdot(q-9,11)+(q-1,13)+7190740931/8]\cdot q^{10}\\
&+[-13899027\cdot(q,2)\cdot(q-1,3)-65556\cdot(q,2)\cdot(q-1,5)+9/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +9/2\cdot(q,2)\cdot(q-3,5)-829/3\cdot(q,2)\cdot(q-1,7)-1/3\cdot(q,2)\cdot(q-2,7)\\
&\quad -1/3\cdot(q,2)\cdot(q-4,7)+175/2\cdot(q,3)\cdot(q-1,4)+8398\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -(q-1,3)\cdot(q,5)+9\cdot(q-1,3)\cdot(q-1,5)-(q-1,3)\cdot(q-2,5)-(q-1,3)\cdot(q-3,5)\\
&\quad -1198045587/4\cdot(q,2)-11639/2\cdot(q,3)+124656021\cdot(q-1,3)+68548663\cdot(q-1,4)\\
&\quad -1059/4\cdot(q,5)+11186612\cdot(q-1,5)-347\cdot(q-2,5)-347\cdot(q-3,5)-85\cdot(q,7)\\
&\quad +1145042\cdot(q-1,7)-71\cdot(q-2,7)-85\cdot(q-3,7)-71\cdot(q-4,7)-85\cdot(q-5,7)\\
&\quad +10267463/8\cdot(q-1,8)+43/2\cdot(q-3,8)-18813/8\cdot(q-5,8)+219539/3\cdot(q-1,9)\\
&\quad -20\cdot(q-2,9)-100/3\cdot(q-4,9)-20\cdot(q-5,9)-100/3\cdot(q-7,9)+3139\cdot(q-1,11)\\
&\quad +40\cdot(q-1,13)+21\cdot(q-1,16)+1663132919/2]\cdot q^{9}\\
&+[-20025121\cdot(q,2)\cdot(q-1,3)-222399\cdot(q,2)\cdot(q-1,5)+10\cdot(q,2)\cdot(q-2,5)\\
&\quad +10\cdot(q,2)\cdot(q-3,5)-2805\cdot(q,2)\cdot(q-1,7)+429/2\cdot(q,3)\cdot(q-1,4)\\
&\quad -1/4\cdot(q,3)\cdot(q-2,5)-1/4\cdot(q,3)\cdot(q-3,5)+40994\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -15/4\cdot(q-1,3)\cdot(q,5)+120\cdot(q-1,3)\cdot(q-1,5)-4\cdot(q-1,3)\cdot(q-2,5)\\
&\quad -4\cdot(q-1,3)\cdot(q-3,5)-2204827671/8\cdot(q,2)-21871/2\cdot(q,3)\\
&\quad +568470307/4\cdot(q-1,3)+309878079/4\cdot(q-1,4)-519\cdot(q,5)+14887241\cdot(q-1,5)\\
&\quad -2359/4\cdot(q-2,5)-2359/4\cdot(q-3,5)-321/2\cdot(q,7)+5924195/3\cdot(q-1,7)\\
&\quad -899/6\cdot(q-2,7)-321/2\cdot(q-3,7)-899/6\cdot(q-4,7)-321/2\cdot(q-5,7)\\
&\quad +37417863/16\cdot(q-1,8)+132\cdot(q-3,8)-53385/16\cdot(q-5,8)+214103\cdot(q-1,9)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -93/2\cdot(q-2,9)-155/2\cdot(q-4,9)-93/2\cdot(q-5,9)-155/2\cdot(q-7,9)+14528\cdot(q-1,11)\\
&\quad +495\cdot(q-1,13)+378\cdot(q-1,16)+5906546797/8]\cdot q^8\\
&+[-26713675\cdot(q,2)\cdot(q-1,3)-590960\cdot(q,2)\cdot(q-1,5)+7/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +7/2\cdot(q,2)\cdot(q-3,5)-16085\cdot(q,2)\cdot(q-1,7)+795/2\cdot(q,3)\cdot(q-1,4)\\
&\quad +145986\cdot(q-1,3)\cdot(q-1,4)-9\cdot(q-1,3)\cdot(q,5)+776\cdot(q-1,3)\cdot(q-1,5)\\
&\quad -9\cdot(q-1,3)\cdot(q-2,5)-9\cdot(q-1,3)\cdot(q-3,5)-1926163555/8\cdot(q,2)-14768\cdot(q,3)\\
&\quad +155939316\cdot(q-1,3)+331077795/4\cdot(q-1,4)-1247/4\cdot(q,5)+18808608\cdot(q-1,5)\\
&\quad -699/2\cdot(q-2,5)-699/2\cdot(q-3,5)-115\cdot(q,7)+3099110\cdot(q-1,7)-99\cdot(q-2,7)\\
&\quad -115\cdot(q-3,7)-99\cdot(q-4,7)-115\cdot(q-5,7)+62024819/16\cdot(q-1,8)+191/2\cdot(q-3,8)\\
&\quad -58117/16\cdot(q-5,8)+1507550/3\cdot(q-1,9)-41\cdot(q-2,9)-205/3\cdot(q-4,9)\\
&\quad -41\cdot(q-5,9)-205/3\cdot(q-7,9)+48275\cdot(q-1,11)+3140\cdot(q-1,13)+2898\cdot(q-1,16)\\
&\quad +4976609959/8]\cdot q^7\\
&+[-32472488\cdot(q,2)\cdot(q-1,3)-1240047\cdot(q,2)\cdot(q-1,5)+5/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +5/2\cdot(q,2)\cdot(q-3,5)-176246/3\cdot(q,2)\cdot(q-1,7)-5/3\cdot(q,2)\cdot(q-2,7)\\
&\quad -5/3\cdot(q,2)\cdot(q-4,7)+497\cdot(q,3)\cdot(q-1,4)-3/4\cdot(q,3)\cdot(q-2,5)\\
&\quad -3/4\cdot(q,3)\cdot(q-3,5)+386705\cdot(q-1,3)\cdot(q-1,4)-79/4\cdot(q-1,3)\cdot(q,5)\\
&\quad +2989\cdot(q-1,3)\cdot(q-1,5)-41/2\cdot(q-1,3)\cdot(q-2,5)-41/2\cdot(q-1,3)\cdot(q-3,5)\\
&\quad -393478813/2\cdot(q,2)-19381\cdot(q,3)+645793243/4\cdot(q-1,3)+82233308\cdot(q-1,4)\\
&\quad -1967/4\cdot(q,5)+22244857\cdot(q-1,5)-2001/4\cdot(q-2,5)-2001/4\cdot(q-3,5)\\
&\quad -183\cdot(q,7)+4355971\cdot(q-1,7)-156\cdot(q-2,7)-183\cdot(q-3,7)-156\cdot(q-4,7)\\
&\quad -183\cdot(q-5,7)+22878373/4\cdot(q-1,8)+597/2\cdot(q-3,8)-10549/4\cdot(q-5,8)\\
&\quad +2841157/3\cdot(q-1,9)-67\cdot(q-2,9)-335/3\cdot(q-4,9)-67\cdot(q-5,9)-335/3\cdot(q-7,9)\\
&\quad +593508/5\cdot(q-1,11)+3/5\cdot(q-3,11)+3/5\cdot(q-4,11)+3/5\cdot(q-5,11)+3/5\cdot(q-9,11)\\
&\quad +12035\cdot(q-1,13)+12831\cdot(q-1,16)+1962067733/4]\cdot q^6\\
&+[-34981138\cdot(q,2)\cdot(q-1,3)-2035793\cdot(q,2)\cdot(q-1,5)-3\cdot(q,2)\cdot(q-2,5)\\
&\quad -3\cdot(q,2)\cdot(q-3,5)-143625\cdot(q,2)\cdot(q-1,7)+749/2\cdot(q,3)\cdot(q-1,4)\\
&\quad +761470\cdot(q-1,3)\cdot(q-1,4)-25\cdot(q-1,3)\cdot(q,5)+7427\cdot(q-1,3)\cdot(q-1,5)\\
&\quad -25\cdot(q-1,3)\cdot(q-2,5)-25\cdot(q-1,3)\cdot(q-3,5)-589772923/4\cdot(q,2)-21971\cdot(q,3)\\
&\quad +152897687\cdot(q-1,3)+74078804\cdot(q-1,4)-941/4\cdot(q,5)+23855470\cdot(q-1,5)\\
&\quad -231\cdot(q-2,5)-231\cdot(q-3,5)-205/2\cdot(q,7)+5325413\cdot(q-1,7)-181/2\cdot(q-2,7)\\
&\quad -205/2\cdot(q-3,7)-181/2\cdot(q-4,7)-205/2\cdot(q-5,7)+58149199/8\cdot(q-1,8)\\
&\quad +361/2\cdot(q-3,8)-557/8\cdot(q-5,8)+4239032/3\cdot(q-1,9)-91/2\cdot(q-2,9)\\
&\quad -455/6\cdot(q-4,9)-91/2\cdot(q-5,9)-455/6\cdot(q-7,9)+1081717/5\cdot(q-1,11)\\
&\quad -3/5\cdot(q-3,11)-3/5\cdot(q-4,11)-3/5\cdot(q-5,11)-3/5\cdot(q-9,11)+29808\cdot(q-1,13)\\
&\quad +36225\cdot(q-1,16)+356007558]\cdot q^5\\
&+[-32012991\cdot(q,2)\cdot(q-1,3)-2548356\cdot(q,2)\cdot(q-1,5)-19/2\cdot(q,2)\cdot(q-2,5)\\
&\quad -19/2\cdot(q,2)\cdot(q-3,5)-238245\cdot(q,2)\cdot(q-1,7)+153/2\cdot(q,3)\cdot(q-1,4)\\
&\quad -3/4\cdot(q,3)\cdot(q-2,5)-3/4\cdot(q,3)\cdot(q-3,5)+1095681\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -179/4\cdot(q-1,3)\cdot(q,5)+12154\cdot(q-1,3)\cdot(q-1,5)-91/2\cdot(q-1,3)\cdot(q-2,5)\\
&\quad -91/2\cdot(q-1,3)\cdot(q-3,5)-394821937/4\cdot(q,2)-22728\cdot(q,3)\\
&\quad +506160503/4\cdot(q-1,3)+58269058\cdot(q-1,4)-555/2\cdot(q,5)+22014870\cdot(q-1,5)\\
&\quad -1059/4\cdot(q-2,5)-1059/4\cdot(q-3,5)-150\cdot(q,7)+5403112\cdot(q-1,7)-133\cdot(q-2,7)\\
&\quad -150\cdot(q-3,7)-133\cdot(q-4,7)-150\cdot(q-5,7)+60858925/8\cdot(q-1,8)+803/2\cdot(q-3,8)\\
&\quad +29905/8\cdot(q-5,8)+4861249/3\cdot(q-1,9)-131/2\cdot(q-2,9)-655/6\cdot(q-4,9)\\
&\quad -131/2\cdot(q-5,9)-655/6\cdot(q-7,9)+286959\cdot(q-1,11)+48795\cdot(q-1,13)\\
&\quad +67347\cdot(q-1,16)+931850685/4]\cdot q^4\\
&+[-23447759\cdot(q,2)\cdot(q-1,3)-2328520\cdot(q,2)\cdot(q-1,5)-11/2\cdot(q,2)\cdot(q-2,5)\\
&\quad -11/2\cdot(q,2)\cdot(q-3,5)-264472\cdot(q,2)\cdot(q-1,7)-3\cdot(q,2)\cdot(q-2,7)\\
&\quad -3\cdot(q,2)\cdot(q-4,7)-487/2\cdot(q,3)\cdot(q-1,4)+1112249\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -27\cdot(q-1,3)\cdot(q,5)+13068\cdot(q-1,3)\cdot(q-1,5)-27\cdot(q-1,3)\cdot(q-2,5)\\
&\quad -27\cdot(q-1,3)\cdot(q-3,5)-454270337/8\cdot(q,2)-38417/2\cdot(q,3)+85878094\cdot(q-1,3)\\
&\quad +151491737/4\cdot(q-1,4)-90\cdot(q,5)+16273260\cdot(q-1,5)-83\cdot(q-2,5)-83\cdot(q-3,5)
\end{aligned}
$$

$$
\begin{aligned}
&-105/2 \cdot (q,7) + 12778063/3 \cdot (q-1,7) - 121/6 \cdot (q-2,7) - 105/2 \cdot (q-3,7) \\
&-121/6 \cdot (q-4,7) - 105/2 \cdot (q-5,7) + 98863409/16 \cdot (q-1,8) + 173 \cdot (q-3,8) \\
&+125169/16 \cdot (q-5,8) + 4090217/3 \cdot (q-1,9) - 26 \cdot (q-2,9) - 130/3 \cdot (q-4,9) \\
&-26 \cdot (q-5,9) - 130/3 \cdot (q-7,9) + 267044 \cdot (q-1,11) + 52380 \cdot (q-1,13) + 82026 \cdot (q-1,16) \\
&+1064914339/8] \cdot q^3 \\
+ [&-12626656 \cdot (q,2) \cdot (q-1,3) - 1448923 \cdot (q,2) \cdot (q-1,5) - 19/2 \cdot (q,2) \cdot (q-2,5) \\
&-19/2 \cdot (q,2) \cdot (q-3,5) - 187664 \cdot (q,2) \cdot (q-1,7) - 647/2 \cdot (q,3) \cdot (q-1,4) \\
&-1/2 \cdot (q,3) \cdot (q-2,5) - 1/2 \cdot (q,3) \cdot (q-3,5) + 748684 \cdot (q-1,3) \cdot (q-1,4) \\
&-44 \cdot (q-1,3) \cdot (q,5) + 8806 \cdot (q-1,3) \cdot (q-1,5) - 89/2 \cdot (q-1,3) \cdot (q-2,5) \\
&-89/2 \cdot (q-1,3) \cdot (q-3,5) - 210385513/8 \cdot (q,2) - 27665/2 \cdot (q,3) \\
&+43709945 \cdot (q-1,3) + 74911189/4 \cdot (q-1,4) - 175/2 \cdot (q,5) + 8763002 \cdot (q-1,5) \\
&-153/2 \cdot (q-2,5) - 153/2 \cdot (q-3,5) - 161/2 \cdot (q,7) + 7156451/3 \cdot (q-1,7) \\
&-449/6 \cdot (q-2,7) - 161/2 \cdot (q-3,7) - 449/6 \cdot (q-4,7) - 161/2 \cdot (q-5,7) \\
&+57411449/16 \cdot (q-1,8) + 593/2 \cdot (q-3,8) + 179953/16 \cdot (q-5,8) + 2349766/3 \cdot (q-1,9) \\
&-40 \cdot (q-2,9) - 200/3 \cdot (q-4,9) - 40 \cdot (q-5,9) - 200/3 \cdot (q-7,9) + 163922 \cdot (q-1,11) \\
&+35400 \cdot (q-1,13) + 62811 \cdot (q-1,16) + 499222175/8] \cdot q^2 \\
+ [&-4361663 \cdot (q,2) \cdot (q-1,3) - 543279 \cdot (q,2) \cdot (q-1,5) - 3/2 \cdot (q,2) \cdot (q-2,5) \\
&-3/2 \cdot (q,2) \cdot (q-3,5) - 76727 \cdot (q,2) \cdot (q-1,7) - 415/2 \cdot (q,3) \cdot (q-1,4) \\
&+297453 \cdot (q-1,3) \cdot (q-1,4) - 10 \cdot (q-1,3) \cdot (q,5) + 3400 \cdot (q-1,3) \cdot (q-1,5) \\
&-10 \cdot (q-1,3) \cdot (q-2,5) - 10 \cdot (q-1,3) \cdot (q-3,5) - 8659979 \cdot (q,2) - 12975/2 \cdot (q,3) \\
&+14495133 \cdot (q-1,3) + 12251165/2 \cdot (q-1,4) - 14 \cdot (q,5) + 2981492 \cdot (q-1,5) \\
&-13 \cdot (q-2,5) - 13 \cdot (q-3,5) - 23/2 \cdot (q,7) + 2482685/3 \cdot (q-1,7) - 29/6 \cdot (q-2,7) \\
&-23/2 \cdot (q-3,7) - 29/6 \cdot (q-4,7) - 23/2 \cdot (q-5,7) + 1309351 \cdot (q-1,8) + 131/2 \cdot (q-3,8) \\
&+26703/2 \cdot (q-5,8) + 271543 \cdot (q-1,9) - 6 \cdot (q-2,9) - 10 \cdot (q-4,9) - 6 \cdot (q-5,9) \\
&-10 \cdot (q-7,9) + 296052/5 \cdot (q-1,11) + 2/5 \cdot (q-3,11) + 2/5 \cdot (q-4,11) + 2/5 \cdot (q-5,11) \\
&+2/5 \cdot (q-9,11) + 13640 \cdot (q-1,13) + 27342 \cdot (q-1,16) + 21094860] \cdot q \\
+ [&-713843 \cdot (q,2) \cdot (q-1,3) - 91821 \cdot (q,2) \cdot (q-1,5) - 5/2 \cdot (q,2) \cdot (q-2,5) \\
&-5/2 \cdot (q,2) \cdot (q-3,5) - 41135/3 \cdot (q,2) \cdot (q-1,7) - 5/3 \cdot (q,2) \cdot (q-2,7) \\
&-5/3 \cdot (q,2) \cdot (q-4,7) - 72 \cdot (q,3) \cdot (q-1,4) - 1/2 \cdot (q,3) \cdot (q-2,5) \\
&-1/2 \cdot (q,3) \cdot (q-3,5) + 52405 \cdot (q-1,3) \cdot (q-1,4) - 31/2 \cdot (q-1,3) \cdot (q,5) \\
&+555 \cdot (q-1,3) \cdot (q-1,5) - 16 \cdot (q-1,3) \cdot (q-2,5) - 16 \cdot (q-1,3) \cdot (q-3,5) \\
&-6091647/4 \cdot (q,2) - 5129/2 \cdot (q,3) + 4611021/2 \cdot (q-1,3) + 1878939/2 \cdot (q-1,4) \\
&-13 \cdot (q,5) + 472663 \cdot (q-1,5) - 10 \cdot (q-2,5) - 10 \cdot (q-3,5) - 41/2 \cdot (q,7) \\
&+395269/3 \cdot (q-1,7) - 31/6 \cdot (q-2,7) - 41/2 \cdot (q-3,7) - 31/6 \cdot (q-4,7) - 41/2 \cdot (q-5,7) \\
&+1855187/8 \cdot (q-1,8) + 91 \cdot (q-3,8) + 114811/8 \cdot (q-5,8) + 127376/3 \cdot (q-1,9) \\
&-11 \cdot (q-2,9) - 55/3 \cdot (q-4,9) - 11 \cdot (q-5,9) - 55/3 \cdot (q-7,9) + 47428/5 \cdot (q-1,11) \\
&-2/5 \cdot (q-3,11) - 2/5 \cdot (q-4,11) - 2/5 \cdot (q-5,11) - 2/5 \cdot (q-9,11) + 2282 \cdot (q-1,13) \\
&+5145 \cdot (q-1,16) + 15350713/4]
\end{aligned}
$$

A.5.11 Dimension $d = 16$

$$
\begin{aligned}
N_{16,5}(q) = {}& q^{130} + q^{129} + 3 \cdot q^{128} + 5 \cdot q^{127} + 10 \cdot q^{126} + 16 \cdot q^{125} + 28 \cdot q^{124} + 43 \cdot q^{123} + 70 \cdot q^{122} \\
&+ 105 \cdot q^{121} + 161 \cdot q^{120} + 235 \cdot q^{119} + 347 \cdot q^{118} + 494 \cdot q^{117} + 707 \cdot q^{116} \\
&+ 986 \cdot q^{115} + 1375 \cdot q^{114} + 1881 \cdot q^{113} + 2569 \cdot q^{112} + 3454 \cdot q^{111} + 4632 \cdot q^{110} \\
&+ 6133 \cdot q^{109} + 8092 \cdot q^{108} + 10566 \cdot q^{107} + 13744 \cdot q^{106} + 17717 \cdot q^{105} \\
&+ 22750 \cdot q^{104} + 28986 \cdot q^{103} + 36785 \cdot q^{102} + 46364 \cdot q^{101} + 58210 \cdot q^{100} \\
&+ 72635 \cdot q^{99} + 90292 \cdot q^{98} + 111622 \cdot q^{97} + 137481 \cdot q^{96} + 168483 \cdot q^{95} \\
&+ 205740 \cdot q^{94} + 250077 \cdot q^{93} + 302929 \cdot q^{92} + 365388 \cdot q^{91} + 439273 \cdot q^{90}
\end{aligned}
$$

$$
\begin{aligned}
&+ 526009 \cdot q^{89} + 627880 \cdot q^{88} + 746708 \cdot q^{87} + 885333 \cdot q^{86} + 1046052 \cdot q^{85} \\
&+ 1232348 \cdot q^{84} + 1447085 \cdot q^{83} + 1694493 \cdot q^{82} + 1978081 \cdot q^{81} \\
&+ 2302942 \cdot q^{80} + 2673326 \cdot q^{79} + 3095291 \cdot q^{78} + 3573931 \cdot q^{77} \\
&+ 4116384 \cdot q^{76} + 4728675 \cdot q^{75} + 5419143 \cdot q^{74} + 6194837 \cdot q^{73} \\
&+ 7065400 \cdot q^{72} + 8039003 \cdot q^{71} + 9126699 \cdot q^{70} + 10337851 \cdot q^{69} \\
&+ [-(q,2) + 11685037] \cdot q^{68} + [-4 \cdot (q,2) + 13178893] \cdot q^{67} \\
&+ [-18 \cdot (q,2) + 14833616] \cdot q^{66} + [-51 \cdot (q,2) + 16661207] \cdot q^{65} \\
&+ [-134 \cdot (q,2) + 18677603] \cdot q^{64} + [-308 \cdot (q,2) + 20896270] \cdot q^{63} \\
&+ [-665 \cdot (q,2) + 23335060] \cdot q^{62} + [-1327 \cdot (q,2) + 26009060] \cdot q^{61} \\
&+ [-2534 \cdot (q,2) + 28938271] \cdot q^{60} + [-4601 \cdot (q,2) + 32139676] \cdot q^{59} \\
&+ [-8065 \cdot (q,2) + 35635799] \cdot q^{58} + [-13644 \cdot (q,2) + 39445947] \cdot q^{57} \\
&+ [-22449 \cdot (q,2) + 43595759] \cdot q^{56} + [-35937 \cdot (q,2) + 48107545] \cdot q^{55} \\
&+ [-56255 \cdot (q,2) + 53010957] \cdot q^{54} + [-86156 \cdot (q,2) + 58332348] \cdot q^{53} \\
&+ [-129511 \cdot (q,2) + 64106701] \cdot q^{52} + [-191222 \cdot (q,2) + 70365936] \cdot q^{51} \\
&+ [-277922 \cdot (q,2) + 77152236] \cdot q^{50} + [-397856 \cdot (q,2) + 84505215] \cdot q^{49} \\
&+ [-561884 \cdot (q,2) + 92476814] \cdot q^{48} + [-783265 \cdot (q,2) + 101117185] \cdot q^{47} \\
&+ [-1079008 \cdot (q,2) + 110491361] \cdot q^{46} + [-1469564 \cdot (q,2) + 120663685] \cdot q^{45} \\
&+ [-1980546 \cdot (q,2) + 131716334] \cdot q^{44} \\
&+ [-2642233 \cdot (q,2) + 3/2 \cdot (q,3) + 3 \cdot (q-1,3) + 287464419/2] \cdot q^{43} \\
&+ [-3491765 \cdot (q,2) + 9/2 \cdot (q,3) + 15 \cdot (q-1,3) + 313630763/2] \cdot q^{42} \\
&+ [-4572278 \cdot (q,2) + 16 \cdot (q,3) + 65 \cdot (q-1,3) + 171071984] \cdot q^{41} \\
&+ [-5935607 \cdot (q,2) + 36 \cdot (q,3) + 209 \cdot (q-1,3) + 186632903] \cdot q^{40} \\
&+ [-7640937 \cdot (q,2) + 167/2 \cdot (q,3) + 605 \cdot (q-1,3) + 407264267/2] \cdot q^{39} \\
&+ [-9757959 \cdot (q,2) + 155 \cdot (q,3) + 1549 \cdot (q-1,3) + (q-1,4) + 222231547] \cdot q^{38} \\
&+ [-12364811 \cdot (q,2) + 585/2 \cdot (q,3) + 3688 \cdot (q-1,3) + 4 \cdot (q-1,4) + 485192839/2] \cdot q^{37} \\
&+ [-15551728 \cdot (q,2) + 476 \cdot (q,3) + 8147 \cdot (q-1,3) + 19 \cdot (q-1,4) + 264922013] \cdot q^{36} \\
&+ [-19417946 \cdot (q,2) + 1555/2 \cdot (q,3) + 17045 \cdot (q-1,3) + 71 \cdot (q-1,4) + 578810607/2] \cdot q^{35} \\
&+ [-24075809 \cdot (q,2) + 2279/2 \cdot (q,3) + 33814 \cdot (q-1,3) + 234 \cdot (q-1,4) + 632547235/2] \cdot q^{34} \\
&+ [-29646373 \cdot (q,2) + 1675 \cdot (q,3) + 64264 \cdot (q-1,3) + 688 \cdot (q-1,4) + 345751243] \cdot q^{33} \\
&+ [-36263813 \cdot (q,2) + 2251 \cdot (q,3) + 117226 \cdot (q-1,3) + 1856 \cdot (q-1,4) + 378089494] \cdot q^{32} \\
&+ [-44069403 \cdot (q,2) + 3039 \cdot (q,3) + 206440 \cdot (q-1,3) + 4613 \cdot (q-1,4) + 413527571] \cdot q^{31} \\
&+ [-53216009 \cdot (q,2) + 7577/2 \cdot (q,3) + 351615 \cdot (q-1,3) + 10721 \cdot (q-1,4) + 904645819/2] \cdot q^{30} \\
&+ [-63859631 \cdot (q,2) + 4765 \cdot (q,3) + 581341 \cdot (q-1,3) + 23446 \cdot (q-1,4) + 494703781] \cdot q^{29} \\
&+ [-76163409 \cdot (q,2) + 11145/2 \cdot (q,3) + 934379 \cdot (q-1,3) + 48599 \cdot (q-1,4) + 1081798481/2] \cdot q^{28} \\
&+ [-90285415 \cdot (q,2) + 13231/2 \cdot (q,3) + 1463530 \cdot (q-1,3) + 96001 \cdot (q-1,4) + 1182159535/2] \cdot q^{27} \\
&+ [-106380652 \cdot (q,2) + 14631/2 \cdot (q,3) + 2236523 \cdot (q-1,3) + 181610 \cdot (q-1,4) + 3/4 \cdot (q,5) \\
&\qquad +(q-1,5) + 2581533595/4] \cdot q^{26} \\
&+ [-124583168 \cdot (q,2) + 16565/2 \cdot (q,3) + 3340163 \cdot (q-1,3) + 330351 \cdot (q-1,4) + 3 \cdot (q,5) \\
&\qquad +16 \cdot (q-1,5) + 1407682343/2] \cdot q^{25} \\
&+ [-145004088 \cdot (q,2) + 8710 \cdot (q,3) + 4879538 \cdot (q-1,3) + 579790 \cdot (q-1,4) + 7 \cdot (q,5) \\
&\qquad +112 \cdot (q-1,5) - 3/2 \cdot (q-2,5) - 3/2 \cdot (q-3,5) + 766395386] \cdot q^{24} \\
&+ [-5 \cdot (q,2) \cdot (q-1,3) - 167705539 \cdot (q,2) + 9471 \cdot (q,3) + 6981171 \cdot (q-1,3) \\
&\qquad +984739 \cdot (q-1,4) + 31/2 \cdot (q,5) + 568 \cdot (q-1,5) - 3/2 \cdot (q-2,5) - 3/2 \cdot (q-3,5) \\
&\qquad +1665609507/2] \cdot q^{23} \\
&+ [-61 \cdot (q,2) \cdot (q-1,3) - 192693554 \cdot (q,2) + 9499 \cdot (q,3) + 9788751 \cdot (q-1,3) \\
&\qquad +1622453 \cdot (q-1,4) + 41/2 \cdot (q,5) + 2262 \cdot (q-1,5) - 15/2 \cdot (q-2,5) - 15/2 \cdot (q-3,5) \\
&\qquad +1805309189/2] \cdot q^{22} \\
&+ [-466 \cdot (q,2) \cdot (q-1,3) - 219882001 \cdot (q,2) + 10005 \cdot (q,3) + 13464187 \cdot (q-1,3)
\end{aligned}
$$

$$
\begin{aligned}
&+2598689\cdot(q-1,4)+129/4\cdot(q,5)+7630\cdot(q-1,5)-15/2\cdot(q-2,5)-15/2\cdot(q-3,5)\\
&+3900953087/4]\cdot q^{21}\\
+\,&[-2492\cdot(q,2)\cdot(q-1,3)-249079742\cdot(q,2)+19283/2\cdot(q,3)+18179402\cdot(q-1,3)\\
&+4053133\cdot(q-1,4)+101/4\cdot(q,5)+22451\cdot(q-1,5)-51/2\cdot(q-2,5)-51/2\cdot(q-3,5)\\
&+(q-1,7)+4198236021/4]\cdot q^{20}\\
+\,&[-10456\cdot(q,2)\cdot(q-1,3)-279944899\cdot(q,2)+10038\cdot(q,3)+24117420\cdot(q-1,3)\\
&+6164727\cdot(q-1,4)+147/4\cdot(q,5)+59199\cdot(q-1,5)-24\cdot(q-2,5)-24\cdot(q-3,5)\\
&+61/3\cdot(q-1,7)+1/3\cdot(q-2,7)+1/3\cdot(q-4,7)+6\cdot(q-1,8)+4496766737/4]\cdot q^{19}\\
+\,&[-36325\cdot(q,2)\cdot(q-1,3)-311967106\cdot(q,2)+9483\cdot(q,3)+31465236\cdot(q-1,3)\\
&+9153572\cdot(q-1,4)+11/4\cdot(q,5)+141963\cdot(q-1,5)-131/2\cdot(q-2,5)-131/2\cdot(q-3,5)\\
&-1/6\cdot(q,7)+517/3\cdot(q-1,7)-1/6\cdot(q-2,7)-1/2\cdot(q-3,7)-1/6\cdot(q-4,7)\\
&-1/2\cdot(q-5,7)+77\cdot(q-1,8)+14367386429/12]\cdot q^{18}\\
+\,&[-108758\cdot(q,2)\cdot(q-1,3)-344412841\cdot(q,2)+19983/2\cdot(q,3)+40423060\cdot(q-1,3)\\
&+13279555\cdot(q-1,4)+15\cdot(q,5)+313616\cdot(q-1,5)-119/2\cdot(q-2,5)-119/2\cdot(q-3,5)\\
&+1/3\cdot(q,7)+955\cdot(q-1,7)+1/2\cdot(q-2,7)-1/2\cdot(q-3,7)+1/2\cdot(q-4,7)-1/2\cdot(q-5,7)\\
&+1101/2\cdot(q-1,8)-1/2\cdot(q-3,8)+7598387557/6]\cdot q^{17}\\
+\,&[-287552\cdot(q,2)\cdot(q-1,3)-376305974\cdot(q,2)+18507/2\cdot(q,3)+51205721\cdot(q-1,3)\\
&+18831556\cdot(q-1,4)-131/2\cdot(q,5)+643338\cdot(q-1,5)-143\cdot(q-2,5)-143\cdot(q-3,5)\\
&-11/3\cdot(q,7)+4012\cdot(q-1,7)-3\cdot(q-2,7)-5\cdot(q-3,7)-3\cdot(q-4,7)-5\cdot(q-5,7)\\
&+5577/2\cdot(q-1,8)-1/2\cdot(q-3,8)+(q-5,8)+3985647485/3]\cdot q^{16}\\
+\,&[-683867\cdot(q,2)\cdot(q-1,3)-812737267/2\cdot(q,2)+9532\cdot(q,3)+64060471\cdot(q-1,3)\\
&+26105680\cdot(q-1,4)-46\cdot(q,5)+1233709\cdot(q-1,5)-249/2\cdot(q-2,5)-249/2\cdot(q-3,5)\\
&-23/6\cdot(q,7)+41594/3\cdot(q-1,7)-7/3\cdot(q-2,7)-5\cdot(q-3,7)-7/3\cdot(q-4,7)\\
&-5\cdot(q-5,7)+43941/4\cdot(q-1,8)-7/2\cdot(q-3,8)-1/4\cdot(q-5,8)+4140179860/3]\cdot q^{15}\\
+\,&[-1481946\cdot(q,2)\cdot(q-1,3)-2\cdot(q,2)\cdot(q-1,5)-433011631\cdot(q,2)+15675/2\cdot(q,3)\\
&+79259201\cdot(q-1,3)+35361508\cdot(q-1,4)-765/4\cdot(q,5)+2221300\cdot(q-1,5)\\
&-268\cdot(q-2,5)-268\cdot(q-3,5)-65/3\cdot(q,7)+123356/3\cdot(q-1,7)-107/6\cdot(q-2,7)\\
&-45/2\cdot(q-3,7)-107/6\cdot(q-4,7)-45/2\cdot(q-5,7)+71637/2\cdot(q-1,8)-5/2\cdot(q-3,8)\\
&-23\cdot(q-5,8)+20/3\cdot(q-1,9)-1/2\cdot(q-2,9)-5/6\cdot(q-4,9)-1/2\cdot(q-5,9)\\
&-5/6\cdot(q-7,9)+17000023001/12]\cdot q^{14}\\
+\,&[-2954967\cdot(q,2)\cdot(q-1,3)-62\cdot(q,2)\cdot(q-1,5)+1/2\cdot(q,3)\cdot(q-1,4)\\
&+(q-1,3)\cdot(q-1,4)-3634396781/8\cdot(q,2)+6915\cdot(q,3)+97079710\cdot(q-1,3)\\
&+187011285/4\cdot(q-1,4)-291/2\cdot(q,5)+3768682\cdot(q-1,5)-449/2\cdot(q-2,5)\\
&-449/2\cdot(q-3,5)-43/2\cdot(q,7)+323326/3\cdot(q-1,7)-47/3\cdot(q-2,7)-22\cdot(q-3,7)\\
&-47/3\cdot(q-4,7)-22\cdot(q-5,7)+1608293/16\cdot(q-1,8)-21/2\cdot(q-3,8)\\
&-2163/16\cdot(q-5,8)+458/3\cdot(q-1,9)-1/2\cdot(q-2,9)-5/6\cdot(q-4,9)-1/2\cdot(q-5,9)\\
&-5/6\cdot(q-7,9)+11468129183/8]\cdot q^{13}\\
+\,&[-5460055\cdot(q,2)\cdot(q-1,3)-779\cdot(q,2)\cdot(q-1,5)+5/2\cdot(q,2)\cdot(q-2,5)\\
&+5/2\cdot(q,2)\cdot(q-3,5)+3/2\cdot(q,3)\cdot(q-1,4)+24\cdot(q-1,3)\cdot(q-1,4)\\
&-936003723/2\cdot(q,2)+3376\cdot(q,3)+117713646\cdot(q-1,3)+120442181/2\cdot(q-1,4)\\
&-713/2\cdot(q,5)+6039847\cdot(q-1,5)-442\cdot(q-2,5)-442\cdot(q-3,5)-389/6\cdot(q,7)\\
&+762430/3\cdot(q-1,7)-170/3\cdot(q-2,7)-65\cdot(q-3,7)-170/3\cdot(q-4,7)-65\cdot(q-5,7)\\
&+999673/4\cdot(q-1,8)-(q-3,8)-1919/4\cdot(q-5,8)+1464\cdot(q-1,9)-6\cdot(q-2,9)\\
&-10\cdot(q-4,9)-6\cdot(q-5,9)-10\cdot(q-7,9)+5\cdot(q-1,11)+8552778137/6]\cdot q^{12}\\
+\,&[-9399287\cdot(q,2)\cdot(q-1,3)-6006\cdot(q,2)\cdot(q-1,5)+2\cdot(q,2)\cdot(q-2,5)\\
&+2\cdot(q,2)\cdot(q-3,5)-(q,2)\cdot(q-1,7)+25/2\cdot(q,3)\cdot(q-1,4)+363\cdot(q-1,3)\cdot(q-1,4)\\
&-3773313755/8\cdot(q,2)+666\cdot(q,3)+141122186\cdot(q-1,3)+301440299/4\cdot(q-1,4)\\
&-1035/4\cdot(q,5)+9165722\cdot(q-1,5)-707/2\cdot(q-2,5)-707/2\cdot(q-3,5)-121/2\cdot(q,7)\\
&+546469\cdot(q-1,7)-99/2\cdot(q-2,7)-121/2\cdot(q-3,7)-99/2\cdot(q-4,7)-121/2\cdot(q-5,7)\\
&+9000379/16\cdot(q-1,8)-12\cdot(q-3,8)-19893/16\cdot(q-5,8)+9047\cdot(q-1,9)-6\cdot(q-2,9)\\
&-10\cdot(q-4,9)-6\cdot(q-5,9)-10\cdot(q-7,9)+111\cdot(q-1,11)+11098988091/8]\cdot q^{11}
\end{aligned}
$$

$$
\begin{aligned}
&+[-15128002\cdot(q,2)\cdot(q-1,3)-32272\cdot(q,2)\cdot(q-1,5)+19/2\cdot(q,2)\cdot(q-2,5)\\
&\quad+19/2\cdot(q,2)\cdot(q-3,5)-51\cdot(q,2)\cdot(q-1,7)+89/2\cdot(q,3)\cdot(q-1,4)\\
&\quad-1/4\cdot(q,3)\cdot(q-2,5)-1/4\cdot(q,3)\cdot(q-3,5)+3156\cdot(q-1,3)\cdot(q-1,4)\\
&\quad-1/4\cdot(q-1,3)\cdot(q,5)+(q-1,3)\cdot(q-1,5)-1/2\cdot(q-1,3)\cdot(q-2,5)\\
&\quad-1/2\cdot(q-1,3)\cdot(q-3,5)-1851358117/4\cdot(q,2)-4849\cdot(q,3)\\
&\quad+667128049/4\cdot(q-1,3)+91263490\cdot(q-1,4)-2091/4\cdot(q,5)+13203608\cdot(q-1,5)\\
&\quad-2515/4\cdot(q-2,5)-2515/4\cdot(q-3,5)-135\cdot(q,7)+1079663\cdot(q-1,7)-120\cdot(q-2,7)\\
&\quad-135\cdot(q-3,7)-120\cdot(q-4,7)-135\cdot(q-5,7)+9305981/8\cdot(q-1,8)+33\cdot(q-3,8)\\
&\quad-20283/8\cdot(q-5,8)+121015/3\cdot(q-1,9)-53/2\cdot(q-2,9)-265/6\cdot(q-4,9)\\
&\quad-53/2\cdot(q-5,9)-265/6\cdot(q-7,9)+1098\cdot(q-1,11)+5\cdot(q-1,13)+3\cdot(q-1,16)\\
&\quad+2629772011/2]\cdot q^{10}\\
&+[-22796871\cdot(q,2)\cdot(q-1,3)-130071\cdot(q,2)\cdot(q-1,5)+9/2\cdot(q,2)\cdot(q-2,5)\\
&\quad+9/2\cdot(q,2)\cdot(q-3,5)-2344/3\cdot(q,2)\cdot(q-1,7)-1/3\cdot(q,2)\cdot(q-2,7)\\
&\quad-1/3\cdot(q,2)\cdot(q-4,7)+305/2\cdot(q,3)\cdot(q-1,4)+18771\cdot(q-1,3)\cdot(q-1,4)\\
&\quad-2\cdot(q-1,3)\cdot(q,5)+27\cdot(q-1,3)\cdot(q-1,5)-2\cdot(q-1,3)\cdot(q-2,5)\\
&\quad-2\cdot(q-1,3)\cdot(q-3,5)-3515387161/8\cdot(q,2)-18291/2\cdot(q,3)+193351518\cdot(q-1,3)\\
&\quad+425577849/4\cdot(q-1,4)-1455/4\cdot(q,5)+18108707\cdot(q-1,5)-923/2\cdot(q-2,5)\\
&\quad-923/2\cdot(q-3,5)-117\cdot(q,7)+5898466/3\cdot(q-1,7)-296/3\cdot(q-2,7)-117\cdot(q-3,7)\\
&\quad-296/3\cdot(q-4,7)-117\cdot(q-5,7)+35602585/16\cdot(q-1,8)+24\cdot(q-3,8)\\
&\quad-66791/16\cdot(q-5,8)+417455/3\cdot(q-1,9)-26\cdot(q-2,9)-130/3\cdot(q-4,9)\\
&\quad-26\cdot(q-5,9)-130/3\cdot(q-7,9)+6651\cdot(q-1,11)+116\cdot(q-1,13)+93\cdot(q-1,16)\\
&\quad+9642502917/8]\cdot q^{9}\\
&+[-32094131\cdot(q,2)\cdot(q-1,3)-408900\cdot(q,2)\cdot(q-1,5)+29/2\cdot(q,2)\cdot(q-2,5)\\
&\quad+29/2\cdot(q,2)\cdot(q-3,5)-6261\cdot(q,2)\cdot(q-1,7)+323\cdot(q,3)\cdot(q-1,4)\\
&\quad-1/4\cdot(q,3)\cdot(q-2,5)-1/4\cdot(q,3)\cdot(q-3,5)+81303\cdot(q-1,3)\cdot(q-1,4)\\
&\quad-19/4\cdot(q-1,3)\cdot(q,5)+275\cdot(q-1,3)\cdot(q-1,5)-5\cdot(q-1,3)\cdot(q-2,5)\\
&\quad-5\cdot(q-1,3)\cdot(q-3,5)-1600941967/4\cdot(q,2)-15699\cdot(q,3)+872648307/4\cdot(q-1,3)\\
&\quad+237031395/2\cdot(q-1,4)-2551/4\cdot(q,5)+23684264\cdot(q-1,5)-2895/4\cdot(q-2,5)\\
&\quad-2895/4\cdot(q-3,5)-411/2\cdot(q,7)+3292211\cdot(q-1,7)-373/2\cdot(q-2,7)\\
&\quad-411/2\cdot(q-3,7)-373/2\cdot(q-4,7)-411/2\cdot(q-5,7)+31419847/8\cdot(q-1,8)\\
&\quad+153\cdot(q-3,8)-44273/8\cdot(q-5,8)+383318\cdot(q-1,9)-123/2\cdot(q-2,9)\\
&\quad-205/2\cdot(q-4,9)-123/2\cdot(q-5,9)-205/2\cdot(q-7,9)+27972\cdot(q-1,11)\\
&\quad+1119\cdot(q-1,13)+1017\cdot(q-1,16)+2118485373/2]\cdot q^{8}\\
&+[-41876578\cdot(q,2)\cdot(q-1,3)-1022818\cdot(q,2)\cdot(q-1,5)+7/2\cdot(q,2)\cdot(q-2,5)\\
&\quad+7/2\cdot(q,2)\cdot(q-3,5)-93382/3\cdot(q,2)\cdot(q-1,7)-1/3\cdot(q,2)\cdot(q-2,7)\\
&\quad-1/3\cdot(q,2)\cdot(q-4,7)+542\cdot(q,3)\cdot(q-1,4)+265075\cdot(q-1,3)\cdot(q-1,4)\\
&\quad-14\cdot(q-1,3)\cdot(q,5)+1520\cdot(q-1,3)\cdot(q-1,5)-14\cdot(q-1,3)\cdot(q-2,5)\\
&\quad-14\cdot(q-1,3)\cdot(q-3,5)-345780890\cdot(q,2)-20549\cdot(q,3)+236590490\cdot(q-1,3)\\
&\quad+124764747\cdot(q-1,4)-1671/4\cdot(q,5)+29433048\cdot(q-1,5)-923/2\cdot(q-2,5)\\
&\quad-923/2\cdot(q-3,5)-160\cdot(q,7)+15082693/3\cdot(q-1,7)-401/3\cdot(q-2,7)-160\cdot(q-3,7)\\
&\quad-401/3\cdot(q-4,7)-160\cdot(q-5,7)+6317166\cdot(q-1,8)+253/2\cdot(q-3,8)\\
&\quad-11299/2\cdot(q-5,8)+2567546/3\cdot(q-1,9)-56\cdot(q-2,9)-280/3\cdot(q-4,9)\\
&\quad-56\cdot(q-5,9)-280/3\cdot(q-7,9)+86521\cdot(q-1,11)+6136\cdot(q-1,13)+6198\cdot(q-1,16)\\
&\quad+3529320951/4]\cdot q^{7}\\
&+[-49803550\cdot(q,2)\cdot(q-1,3)-2042002\cdot(q,2)\cdot(q-1,5)+9/2\cdot(q,2)\cdot(q-2,5)\\
&\quad+9/2\cdot(q,2)\cdot(q-3,5)-309941/3\cdot(q,2)\cdot(q-1,7)-5/3\cdot(q,2)\cdot(q-2,7)\\
&\quad-5/3\cdot(q,2)\cdot(q-4,7)+1259/2\cdot(q,3)\cdot(q-1,4)-3/4\cdot(q,3)\cdot(q-2,5)\\
&\quad-3/4\cdot(q,3)\cdot(q-3,5)+656047\cdot(q-1,3)\cdot(q-1,4)-99/4\cdot(q-1,3)\cdot(q,5)\\
&\quad+5320\cdot(q-1,3)\cdot(q-1,5)-51/2\cdot(q-1,3)\cdot(q-2,5)-51/2\cdot(q-1,3)\cdot(q-3,5)\\
&\quad-2233119759/8\cdot(q,2)-26138\cdot(q,3)+966699403/4\cdot(q-1,3)\\
&\quad+488254043/4\cdot(q-1,4)-2317/4\cdot(q,5)+34222136\cdot(q-1,5)-2365/4\cdot(q-2,5)\\
&\quad-2365/4\cdot(q-3,5)-449/2\cdot(q,7)+6885406\cdot(q-1,7)-383/2\cdot(q-2,7)
\end{aligned}
$$

$$
\begin{aligned}
&-449/2\cdot(q-3,7)-383/2\cdot(q-4,7)-449/2\cdot(q-5,7)+144855583/16\cdot(q-1,8)\\
&+357\cdot(q-3,8)-59073/16\cdot(q-5,8)+4638137/3\cdot(q-1,9)-86\cdot(q-2,9)\\
&-430/3\cdot(q-4,9)-86\cdot(q-5,9)-430/3\cdot(q-7,9)+201048\cdot(q-1,11)+21379\cdot(q-1,13)\\
&+23982\cdot(q-1,16)+5495832339/8]\cdot q^6\\
+\,&[-52485021\cdot(q,2)\cdot(q-1,3)-3214705\cdot(q,2)\cdot(q-1,5)-3\cdot(q,2)\cdot(q-2,5)\\
&-3\cdot(q,2)\cdot(q-3,5)-235935\cdot(q,2)\cdot(q-1,7)+438\cdot(q,3)\cdot(q-1,4)\\
&+1224356\cdot(q-1,3)\cdot(q-1,4)-35\cdot(q-1,3)\cdot(q,5)+12349\cdot(q-1,3)\cdot(q-1,5)\\
&-35\cdot(q-1,3)\cdot(q-2,5)-35\cdot(q-1,3)\cdot(q-3,5)-412931489/2\cdot(q,2)-28750\cdot(q,3)\\
&+225385493\cdot(q-1,3)+108193266\cdot(q-1,4)-1253/4\cdot(q,5)+36026275\cdot(q-1,5)\\
&-309\cdot(q-2,5)-309\cdot(q-3,5)-144\cdot(q,7)+24622783/3\cdot(q-1,7)-377/3\cdot(q-2,7)\\
&-144\cdot(q-3,7)-377/3\cdot(q-4,7)-144\cdot(q-5,7)+44744185/4\cdot(q-1,8)+241\cdot(q-3,8)\\
&+1877/4\cdot(q-5,8)+2224310\cdot(q-1,9)-129/2\cdot(q-2,9)-215/2\cdot(q-4,9)\\
&-129/2\cdot(q-5,9)-215/2\cdot(q-7,9)+350071\cdot(q-1,11)+49536\cdot(q-1,13)\\
&+61929\cdot(q-1,16)+1967147963/4]\cdot q^5\\
+\,&[-46987227\cdot(q,2)\cdot(q-1,3)-3883697\cdot(q,2)\cdot(q-1,5)-13\cdot(q,2)\cdot(q-2,5)\\
&-13\cdot(q,2)\cdot(q-3,5)-372099\cdot(q,2)\cdot(q-1,7)-(q,2)\cdot(q-2,7)-(q,2)\cdot(q-4,7)\\
&+54\cdot(q,3)\cdot(q-1,4)-3/4\cdot(q,3)\cdot(q-2,5)-3/4\cdot(q,3)\cdot(q-3,5)\\
&+1687610\cdot(q-1,3)\cdot(q-1,4)-219/4\cdot(q-1,3)\cdot(q,5)+19257\cdot(q-1,3)\cdot(q-1,5)\\
&-111/2\cdot(q-1,3)\cdot(q-2,5)-111/2\cdot(q-1,3)\cdot(q-3,5)-136287199\cdot(q,2)\\
&-58029/2\cdot(q,3)+733554291/4\cdot(q-1,3)+83667729\cdot(q-1,4)-599/2\cdot(q,5)\\
&+32594602\cdot(q-1,5)-1133/4\cdot(q-2,5)-1133/4\cdot(q-3,5)-172\cdot(q,7)\\
&+8125939\cdot(q-1,7)-142\cdot(q-2,7)-172\cdot(q-3,7)-142\cdot(q-4,7)-172\cdot(q-5,7)\\
&+22794137/2\cdot(q-1,8)+969/2\cdot(q-3,8)+6258\cdot(q-5,8)+7418017/3\cdot(q-1,9)\\
&-155/2\cdot(q-2,9)-775/6\cdot(q-4,9)-155/2\cdot(q-5,9)-775/6\cdot(q-7,9)\\
&+447483\cdot(q-1,11)+77247\cdot(q-1,13)+108108\cdot(q-1,16)+317187467]\cdot q^4\\
+\,&[-33684641\cdot(q,2)\cdot(q-1,3)-3444296\cdot(q,2)\cdot(q-1,5)-11/2\cdot(q,2)\cdot(q-2,5)\\
&-11/2\cdot(q,2)\cdot(q-3,5)-397497\cdot(q,2)\cdot(q-1,7)-3\cdot(q,2)\cdot(q-2,7)\\
&-3\cdot(q,2)\cdot(q-4,7)-671/2\cdot(q,3)\cdot(q-1,4)+1654855\cdot(q-1,3)\cdot(q-1,4)\\
&-36\cdot(q-1,3)\cdot(q,5)+19956\cdot(q-1,3)\cdot(q-1,5)-36\cdot(q-1,3)\cdot(q-2,5)\\
&-36\cdot(q-1,3)\cdot(q-3,5)-617983833/8\cdot(q,2)-47875/2\cdot(q,3)+122254353\cdot(q-1,3)\\
&+213738033/4\cdot(q-1,4)-247/2\cdot(q,5)+23617193\cdot(q-1,5)-233/2\cdot(q-2,5)\\
&-233/2\cdot(q-3,5)-149/2\cdot(q,7)+6260535\cdot(q-1,7)-77/2\cdot(q-2,7)-149/2\cdot(q-3,7)\\
&-77/2\cdot(q-4,7)-149/2\cdot(q-5,7)+144465697/16\cdot(q-1,8)+229\cdot(q-3,8)\\
&+193761/16\cdot(q-5,8)+6079649/3\cdot(q-1,9)-38\cdot(q-2,9)-190/3\cdot(q-4,9)\\
&-38\cdot(q-5,9)-190/3\cdot(q-7,9)+404240\cdot(q-1,11)+79968\cdot(q-1,13)\\
&+125730\cdot(q-1,16)+1428772079/8]\cdot q^3\\
+\,&[-17775308\cdot(q,2)\cdot(q-1,3)-2090880\cdot(q,2)\cdot(q-1,5)-14\cdot(q,2)\cdot(q-2,5)\\
&-14\cdot(q,2)\cdot(q-3,5)-273796\cdot(q,2)\cdot(q-1,7)-412\cdot(q,3)\cdot(q-1,4)\\
&-1/2\cdot(q,3)\cdot(q-2,5)-1/2\cdot(q,3)\cdot(q-3,5)+1083427\cdot(q-1,3)\cdot(q-1,4)\\
&-53\cdot(q-1,3)\cdot(q,5)+13090\cdot(q-1,3)\cdot(q-1,5)-107/2\cdot(q-1,3)\cdot(q-2,5)\\
&-107/2\cdot(q-1,3)\cdot(q-3,5)-281821251/8\cdot(q,2)-16904\cdot(q,3)+61127058\cdot(q-1,3)\\
&+103836641/4\cdot(q-1,4)-149/2\cdot(q,5)+12480485\cdot(q-1,5)-59\cdot(q-2,5)\\
&-59\cdot(q-3,5)-171/2\cdot(q,7)+3435052\cdot(q-1,7)-155/2\cdot(q-2,7)-171/2\cdot(q-3,7)\\
&-155/2\cdot(q-4,7)-171/2\cdot(q-5,7)+82037243/16\cdot(q-1,8)+717/2\cdot(q-3,8)\\
&+270211/16\cdot(q-5,8)+3417766/3\cdot(q-1,9)-43\cdot(q-2,9)-215/3\cdot(q-4,9)\\
&-43\cdot(q-5,9)-215/3\cdot(q-7,9)+242338\cdot(q-1,11)+52572\cdot(q-1,13)+92997\cdot(q-1,16)\\
&+660189353/8]\cdot q^2\\
+\,&[-6027284\cdot(q,2)\cdot(q-1,3)-768346\cdot(q,2)\cdot(q-1,5)-3/2\cdot(q,2)\cdot(q-2,5)\\
&-3/2\cdot(q,2)\cdot(q-3,5)-328103/3\cdot(q,2)\cdot(q-1,7)-2/3\cdot(q,2)\cdot(q-2,7)\\
&-2/3\cdot(q,2)\cdot(q-4,7)-517/2\cdot(q,3)\cdot(q-1,4)+421047\cdot(q-1,3)\cdot(q-1,4)\\
&-13\cdot(q-1,3)\cdot(q,5)+4944\cdot(q-1,3)\cdot(q-1,5)-13\cdot(q-1,3)\cdot(q-2,5)\\
&-13\cdot(q-1,3)\cdot(q-3,5)-45676125/4\cdot(q,2)-15609/2\cdot(q,3)+19931018\cdot(q-1,3)
\end{aligned}
$$

$$
\begin{aligned}
&+8342296 \cdot (q-1,4) - 41/2 \cdot (q,5) + 4175571 \cdot (q-1,5) - 39/2 \cdot (q-2,5) - 39/2 \cdot (q-3,5)\\
&-33/2 \cdot (q,7) + 3512312/3 \cdot (q-1,7) - 23/6 \cdot (q-2,7) - 33/2 \cdot (q-3,7) - 23/6 \cdot (q-4,7)\\
&-33/2 \cdot (q-5,7) + 14682337/8 \cdot (q-1,8) + 86 \cdot (q-3,8) + 157753/8 \cdot (q-5,8)\\
&+388063 \cdot (q-1,9) - 9 \cdot (q-2,9) - 15 \cdot (q-4,9) - 9 \cdot (q-5,9) - 15 \cdot (q-7,9)\\
&+85910 \cdot (q-1,11) + 19828 \cdot (q-1,13) + 39426 \cdot (q-1,16) + 110044961/4] \cdot q\\
+ &[-970249 \cdot (q,2) \cdot (q-1,3) - 127772 \cdot (q,2) \cdot (q-1,5) - 4 \cdot (q,2) \cdot (q-2,5)\\
&-4 \cdot (q,2) \cdot (q-3,5) - 57566/3 \cdot (q,2) \cdot (q-1,7) - 5/3 \cdot (q,2) \cdot (q-2,7)\\
&-5/3 \cdot (q,2) \cdot (q-4,7) - 89 \cdot (q,3) \cdot (q-1,4) - 1/2 \cdot (q,3) \cdot (q-2,5)\\
&-1/2 \cdot (q,3) \cdot (q-3,5) + 72901 \cdot (q-1,3) \cdot (q-1,4) - 37/2 \cdot (q-1,3) \cdot (q,5)\\
&+797 \cdot (q-1,3) \cdot (q-1,5) - 19 \cdot (q-1,3) \cdot (q-2,5) - 19 \cdot (q-1,3) \cdot (q-3,5)\\
&-7914461/4 \cdot (q,2) - 6065/2 \cdot (q,3) + 6242381/2 \cdot (q-1,3) + 2504965/2 \cdot (q-1,4)\\
&-11/2 \cdot (q,5) + 652475 \cdot (q-1,5) - (q-2,5) - (q-3,5) - 41/2 \cdot (q,7) + 550949/3 \cdot (q-1,7)\\
&-23/6 \cdot (q-2,7) - 41/2 \cdot (q-3,7) - 23/6 \cdot (q-4,7) - 41/2 \cdot (q-5,7)\\
&+2565865/8 \cdot (q-1,8) + 110 \cdot (q-3,8) + 168345/8 \cdot (q-5,8) + 179507/3 \cdot (q-1,9)\\
&-11 \cdot (q-2,9) - 55/3 \cdot (q-4,9) - 11 \cdot (q-5,9) - 55/3 \cdot (q-7,9) + 13560 \cdot (q-1,11)\\
&+3262 \cdot (q-1,13) + 7269 \cdot (q-1,16) + 19777889/4]
\end{aligned}
$$

A.5.12 Dimension $d = 17$

$$
\begin{aligned}
N_{17,5}(q) = &\, q^{132} + q^{131} + 3 \cdot q^{130} + 5 \cdot q^{129} + 10 \cdot q^{128} + 16 \cdot q^{127} + 28 \cdot q^{126} + 43 \cdot q^{125} + 70 \cdot q^{124}\\
&+ 105 \cdot q^{123} + 161 \cdot q^{122} + 235 \cdot q^{121} + 348 \cdot q^{120} + 495 \cdot q^{119} + 709 \cdot q^{118}\\
&+ 990 \cdot q^{117} + 1382 \cdot q^{116} + 1892 \cdot q^{115} + 2587 \cdot q^{114} + 3481 \cdot q^{113} + 4674 \cdot q^{112}\\
&+ 6195 \cdot q^{111} + 8183 \cdot q^{110} + 10696 \cdot q^{109} + 13931 \cdot q^{108} + 17977 \cdot q^{107}\\
&+ 23112 \cdot q^{106} + 29482 \cdot q^{105} + 37460 \cdot q^{104} + 47270 \cdot q^{103} + 59422 \cdot q^{102}\\
&+ 74235 \cdot q^{101} + 92397 \cdot q^{100} + 114363 \cdot q^{99} + 141032 \cdot q^{98} + 173046 \cdot q^{97}\\
&+ 211579 \cdot q^{96} + 257488 \cdot q^{95} + 312297 \cdot q^{94} + 377153 \cdot q^{93} + 453983 \cdot q^{92}\\
&+ 544293 \cdot q^{91} + 650517 \cdot q^{90} + 774579 \cdot q^{89} + 919520 \cdot q^{88} + 1087780 \cdot q^{87}\\
&+ 1283088 \cdot q^{86} + 1508509 \cdot q^{85} + 1768591 \cdot q^{84} + 2067086 \cdot q^{83}\\
&+ 2409499 \cdot q^{82} + 2800402 \cdot q^{81} + 3246345 \cdot q^{80} + 3752836 \cdot q^{79}\\
&+ 4327628 \cdot q^{78} + 4977245 \cdot q^{77} + 5710783 \cdot q^{76} + 6535909 \cdot q^{75}\\
&+ 7463155 \cdot q^{74} + 8501459 \cdot q^{73} + 9662939 \cdot q^{72} + 10957844 \cdot q^{71}\\
&+ 12400021 \cdot q^{70} + 14001164 \cdot q^{69} + [-4 \cdot (q,2) + 15776908] \cdot q^{68}\\
&+ [-14 \cdot (q,2) + 17740488] \cdot q^{67} + [-46 \cdot (q,2) + 19909491] \cdot q^{66}\\
&+ [-121 \cdot (q,2) + 22298755] \cdot q^{65} + [-292 \cdot (q,2) + 24927979] \cdot q^{64}\\
&+ [-633 \cdot (q,2) + 27813767] \cdot q^{63} + [-1296 \cdot (q,2) + 30978092] \cdot q^{62}\\
&+ [-2490 \cdot (q,2) + 34439577] \cdot q^{61} + [-4586 \cdot (q,2) + 38222824] \cdot q^{60}\\
&+ [-8088 \cdot (q,2) + 42348754] \cdot q^{59} + [-13808 \cdot (q,2) + 46845134] \cdot q^{58}\\
&+ [-22833 \cdot (q,2) + 51735841] \cdot q^{57} + [-36797 \cdot (q,2) + 57052497] \cdot q^{56}\\
&+ [-57838 \cdot (q,2) + 62822848] \cdot q^{55} + [-89027 \cdot (q,2) + 69083638] \cdot q^{54}\\
&+ [-134298 \cdot (q,2) + 75867779] \cdot q^{53} + [-199077 \cdot (q,2) + 83218846] \cdot q^{52}\\
&+ [-290199 \cdot (q,2) + 91176957] \cdot q^{51} + [-416783 \cdot (q,2) + 99794817] \cdot q^{50}\\
&+ [-590117 \cdot (q,2) + 109122496] \cdot q^{49} + [-824870 \cdot (q,2) + 119225136] \cdot q^{48}\\
&+ [-1138874 \cdot (q,2) + 130166197] \cdot q^{47} + [-1554757 \cdot (q,2) + 142027378] \cdot q^{46}\\
&+ [-2099591 \cdot (q,2) + 154890154] \cdot q^{45}\\
&+ [-2806951 \cdot (q,2) + 1/2 \cdot (q,3) + (q-1,3) + 337715165/2] \cdot q^{44}\\
&+ [-3716341 \cdot (q,2) + 4 \cdot (q,3) + 10 \cdot (q-1,3) + 184034334] \cdot q^{43}
\end{aligned}
$$

$$
\begin{aligned}
&+ [-4875782 \cdot (q,2) + 21/2 \cdot (q,3) + 41 \cdot (q-1,3) + 401101127/2] \cdot q^{42} \\
&+ [-6340778 \cdot (q,2) + 61/2 \cdot (q,3) + 147 \cdot (q-1,3) + 437079003/2] \cdot q^{41} \\
&+ [-8177500 \cdot (q,2) + 63 \cdot (q,3) + 431 \cdot (q-1,3) + 238163956] \cdot q^{40} \\
&+ [-10461118 \cdot (q,2) + 271/2 \cdot (q,3) + 1158 \cdot (q-1,3) + 519181931/2] \cdot q^{39} \\
&+ [-13279490 \cdot (q,2) + 481/2 \cdot (q,3) + 2812 \cdot (q-1,3) + 2 \cdot (q-1,4) + 566040609/2] \cdot q^{38} \\
&+ [-16730653 \cdot (q,2) + 432 \cdot (q,3) + 6400 \cdot (q-1,3) + 10 \cdot (q-1,4) + 308656139] \cdot q^{37} \\
&+ [-20927025 \cdot (q,2) + 1359/2 \cdot (q,3) + 13640 \cdot (q-1,3) + 42 \cdot (q-1,4) + 673474585/2] \cdot q^{36} \\
&+ [-25991614 \cdot (q,2) + 2151/2 \cdot (q,3) + 27663 \cdot (q-1,3) + 144 \cdot (q-1,4) + 735008595/2] \cdot q^{35} \\
&+ [-32062764 \cdot (q,2) + 3073/2 \cdot (q,3) + 53452 \cdot (q-1,3) + 445 \cdot (q-1,4) + 802463907/2] \cdot q^{34} \\
&+ [-39288782 \cdot (q,2) + 4409/2 \cdot (q,3) + 99239 \cdot (q-1,3) + 1240 \cdot (q-1,4) + 876381073/2] \cdot q^{33} \\
&+ [-47832935 \cdot (q,2) + 2902 \cdot (q,3) + 177354 \cdot (q-1,3) + 3193 \cdot (q-1,4) + 478679204] \cdot q^{32} \\
&+ [-57866156 \cdot (q,2) + 3845 \cdot (q,3) + 306628 \cdot (q-1,3) + 7646 \cdot (q-1,4) + 522980821] \cdot q^{31} \\
&+ [-69572014 \cdot (q,2) + 4716 \cdot (q,3) + 513704 \cdot (q-1,3) + 17208 \cdot (q-1,4) + 571395170] \cdot q^{30} \\
&+ [-83136425 \cdot (q,2) + 11695/2 \cdot (q,3) + 836642 \cdot (q-1,3) + 36612 \cdot (q-1,4) + 1248367775/2] \cdot q^{29} \\
&+ [-98751786 \cdot (q,2) + 6746 \cdot (q,3) + 1326420 \cdot (q-1,3) + 74092 \cdot (q-1,4) + 681602274] \cdot q^{28} \\
&+ [-116602173 \cdot (q,2) + 7924 \cdot (q,3) + 2051559 \cdot (q-1,3) + 143324 \cdot (q-1,4) + 743830657] \cdot q^{27} \\
&+ [-136864614 \cdot (q,2) + 17335/2 \cdot (q,3) + 3098939 \cdot (q-1,3) + 266131 \cdot (q-1,4) + 3/4 \cdot (q,5) \\
&\quad +3 \cdot (q-1,5) - 1/2 \cdot (q-2,5) - 1/2 \cdot (q-3,5) + 3244000063/4] \cdot q^{26} \\
&+ [-159687567 \cdot (q,2) + 9734 \cdot (q,3) + 4578570 \cdot (q-1,3) + 476120 \cdot (q-1,4) + 15/4 \cdot (q,5) \\
&\quad +36 \cdot (q-1,5) - 1/2 \cdot (q-2,5) - 1/2 \cdot (q-3,5) + 3532424309/4] \cdot q^{25} \\
&+ [-(q,2) \cdot (q-1,3) - 185187142 \cdot (q,2) + 10138 \cdot (q,3) + 6622134 \cdot (q-1,3) \\
&\quad +823197 \cdot (q-1,4) + 35/4 \cdot (q,5) + 220 \cdot (q-1,5) - 3 \cdot (q-2,5) - 3 \cdot (q-3,5) + 3840099049/4] \cdot q^{24} \\
&+ [-15 \cdot (q,2) \cdot (q-1,3) - 213416003 \cdot (q,2) + 10962 \cdot (q,3) + 9386350 \cdot (q-1,3) \\
&\quad +1379266 \cdot (q-1,4) + 77/4 \cdot (q,5) + 1018 \cdot (q-1,5) - 3 \cdot (q-2,5) - 3 \cdot (q-3,5) \\
&\quad +4165612311/4] \cdot q^{23} \\
&+ [-134 \cdot (q,2) \cdot (q-1,3) - 244353224 \cdot (q,2) + 10904 \cdot (q,3) + 13047163 \cdot (q-1,3) \\
&\quad +2244378 \cdot (q-1,4) + 93/4 \cdot (q,5) + 3805 \cdot (q-1,5) - 12 \cdot (q-2,5) - 12 \cdot (q-3,5) \\
&\quad +4506656655/4] \cdot q^{22} \\
&+ [-874 \cdot (q,2) \cdot (q-1,3) - 277862141 \cdot (q,2) + 11456 \cdot (q,3) + 17800694 \cdot (q-1,3) \\
&\quad +3553902 \cdot (q-1,4) + 73/2 \cdot (q,5) + 12218 \cdot (q-1,5) - 23/2 \cdot (q-2,5) - 23/2 \cdot (q-3,5) \\
&\quad +2429745299/2] \cdot q^{21} \\
&+ [-4268 \cdot (q,2) \cdot (q-1,3) - 313673223 \cdot (q,2) + 21943/2 \cdot (q,3) + 23853071 \cdot (q-1,3) \\
&\quad +5484515 \cdot (q-1,4) + 101/4 \cdot (q,5) + 34577 \cdot (q-1,5) - 69/2 \cdot (q-2,5) - 69/2 \cdot (q-3,5) \\
&\quad +3 \cdot (q-1,7) + (q-1,8) + 5218917457/4] \cdot q^{20} \\
&+ [-16809 \cdot (q,2) \cdot (q-1,3) - 351331538 \cdot (q,2) + 11438 \cdot (q,3) + 31422231 \cdot (q-1,3) \\
&\quad +8259963 \cdot (q-1,4) + 38 \cdot (q,5) + 88285 \cdot (q-1,5) - 32 \cdot (q-2,5) - 32 \cdot (q-3,5) + 41 \cdot (q-1,7) \\
&\quad +17 \cdot (q-1,8) + 1394416407] \cdot q^{19} \\
&+ [-55750 \cdot (q,2) \cdot (q-1,3) - 390174075 \cdot (q,2) + 21521/2 \cdot (q,3) + 40730113 \cdot (q-1,3) \\
&\quad +12152094 \cdot (q-1,4) - 6 \cdot (q,5) + 206061 \cdot (q-1,5) - 169/2 \cdot (q-2,5) - 169/2 \cdot (q-3,5) \\
&\quad -2/3 \cdot (q,7) + 901/3 \cdot (q-1,7) - 2/3 \cdot (q-2,7) - (q-3,7) - 2/3 \cdot (q-4,7) - (q-5,7) \\
&\quad +157 \cdot (q-1,8) + 8889542629/6] \cdot q^{18} \\
&+ [-160939 \cdot (q,2) \cdot (q-1,3) - 858535829/2 \cdot (q,2) + 11354 \cdot (q,3) + 52015244 \cdot (q-1,3) \\
&\quad +17477694 \cdot (q-1,4) + 39/4 \cdot (q,5) + 444732 \cdot (q-1,5) - 149/2 \cdot (q-2,5) - 149/2 \cdot (q-3,5) \\
&\quad +4630/3 \cdot (q-1,7) - 2/3 \cdot (q-2,7) - (q-3,7) - 2/3 \cdot (q-4,7) - (q-5,7) + 3901/4 \cdot (q-1,8) \\
&\quad -1/2 \cdot (q-3,8) + 3/4 \cdot (q-5,8) + 6252875575/4] \cdot q^{17} \\
&+ [-413120 \cdot (q,2) \cdot (q-1,3) - 467385838 \cdot (q,2) + 10423 \cdot (q,3) + 65535178 \cdot (q-1,3) \\
&\quad +24583098 \cdot (q-1,4) - 177/2 \cdot (q,5) + 893981 \cdot (q-1,5) - 349/2 \cdot (q-2,5) \\
&\quad -349/2 \cdot (q-3,5) - 37/6 \cdot (q,7) + 18506/3 \cdot (q-1,7) - 41/6 \cdot (q-2,7) - 15/2 \cdot (q-3,7) \\
&\quad -41/6 \cdot (q-4,7) - 15/2 \cdot (q-5,7) + 4525 \cdot (q-1,8) - 1/2 \cdot (q-3,8) + 5/2 \cdot (q-5,8)
\end{aligned}
$$

$$
\begin{aligned}
&\quad +4906806182/3]\cdot q^{16}\\
&+[-958548\cdot(q,2)\cdot(q-1,3)-502941759\cdot(q,2)+10678\cdot(q,3)+81587425\cdot(q-1,3)\\
&\quad +33815128\cdot(q-1,4)-61\cdot(q,5)+1683895\cdot(q-1,5)-297/2\cdot(q-2,5)-297/2\cdot(q-3,5)\\
&\quad -19/3\cdot(q,7)+61633/3\cdot(q-1,7)-31/6\cdot(q-2,7)-15/2\cdot(q-3,7)-31/6\cdot(q-4,7)\\
&\quad -15/2\cdot(q-5,7)+16840\cdot(q-1,8)-7/2\cdot(q-3,8)+1/2\cdot(q-5,8)+5082725794/3]\cdot q^{15}\\
&+[-2033869\cdot(q,2)\cdot(q-1,3)-5\cdot(q,2)\cdot(q-1,5)+1/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +1/2\cdot(q,2)\cdot(q-3,5)-4271841625/8\cdot(q,2)+17161/2\cdot(q,3)+100496070\cdot(q-1,3)\\
&\quad +181857995/4\cdot(q-1,4)-465/2\cdot(q,5)+2983784\cdot(q-1,5)-641/2\cdot(q-2,5)\\
&\quad -641/2\cdot(q-3,5)-86/3\cdot(q,7)+59143\cdot(q-1,7)-55/2\cdot(q-2,7)-59/2\cdot(q-3,7)\\
&\quad -55/2\cdot(q-4,7)-59/2\cdot(q-5,7)+842169/16\cdot(q-1,8)-2\cdot(q-3,8)-631/16\cdot(q-5,8)\\
&\quad +40/3\cdot(q-1,9)-(q-2,9)-5/3\cdot(q-4,9)-(q-5,9)-5/3\cdot(q-7,9)+41615170465/24]\cdot q^{14}\\
&+[-3981807\cdot(q,2)\cdot(q-1,3)-122\cdot(q,2)\cdot(q-1,5)+1/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +1/2\cdot(q,2)\cdot(q-3,5)+1/2\cdot(q,3)\cdot(q-1,4)+2\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -4465170471/8\cdot(q,2)+7404\cdot(q,3)+122581905\cdot(q-1,3)+238714387/4\cdot(q-1,4)\\
&\quad -683/4\cdot(q,5)+4990079\cdot(q-1,5)-521/2\cdot(q-2,5)-521/2\cdot(q-3,5)-28\cdot(q,7)\\
&\quad +453458/3\cdot(q-1,7)-149/6\cdot(q-2,7)-57/2\cdot(q-3,7)-149/6\cdot(q-4,7)\\
&\quad -57/2\cdot(q-5,7)+2287399/16\cdot(q-1,8)-10\cdot(q-3,8)-3449/16\cdot(q-5,8)\\
&\quad +790/3\cdot(q-1,9)-(q-2,9)-5/3\cdot(q-4,9)-(q-5,9)-5/3\cdot(q-7,9)+13991567443/8]\cdot q^{13}\\
&+[-7239332\cdot(q,2)\cdot(q-1,3)-1327\cdot(q,2)\cdot(q-1,5)+7/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +7/2\cdot(q,2)\cdot(q-3,5)+5/2\cdot(q,3)\cdot(q-1,4)+50\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -4582026665/8\cdot(q,2)+3211\cdot(q,3)+148041274\cdot(q-1,3)+305321589/4\cdot(q-1,4)\\
&\quad -1665/4\cdot(q,5)+7894400\cdot(q-1,5)-513\cdot(q-2,5)-513\cdot(q-3,5)-238/3\cdot(q,7)\\
&\quad +348730\cdot(q-1,7)-145/2\cdot(q-2,7)-159/2\cdot(q-3,7)-145/2\cdot(q-4,7)\\
&\quad -159/2\cdot(q-5,7)+5539665/16\cdot(q-1,8)+3\cdot(q-3,8)-11503/16\cdot(q-5,8)\\
&\quad +6922/3\cdot(q-1,9)-17/2\cdot(q-2,9)-85/6\cdot(q-4,9)-17/2\cdot(q-5,9)-85/6\cdot(q-7,9)\\
&\quad +10\cdot(q-1,11)+41594271527/24]\cdot q^{12}\\
&+[-12283584\cdot(q,2)\cdot(q-1,3)-9328\cdot(q,2)\cdot(q-1,5)+3\cdot(q,2)\cdot(q-2,5)\\
&\quad +3\cdot(q,2)\cdot(q-3,5)-3\cdot(q,2)\cdot(q-1,7)+37/2\cdot(q,3)\cdot(q-1,4)+637\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -1149776085/2\cdot(q,2)+23\cdot(q,3)+176762639\cdot(q-1,3)+189713573/2\cdot(q-1,4)\\
&\quad -581/2\cdot(q,5)+11841452\cdot(q-1,5)-399\cdot(q-2,5)-399\cdot(q-3,5)-145/2\cdot(q,7)\\
&\quad +735563\cdot(q-1,7)-133/2\cdot(q-2,7)-145/2\cdot(q-3,7)-133/2\cdot(q-4,7)\\
&\quad -145/2\cdot(q-5,7)+3050081/4\cdot(q-1,8)-9\cdot(q-3,8)-7087/4\cdot(q-5,8)\\
&\quad +40183/3\cdot(q-1,9)-17/2\cdot(q-2,9)-85/6\cdot(q-4,9)-17/2\cdot(q-5,9)-85/6\cdot(q-7,9)\\
&\quad +187\cdot(q-1,11)+3360893417/2]\cdot q^{11}\\
&+[-19514615\cdot(q,2)\cdot(q-1,3)-47075\cdot(q,2)\cdot(q-1,5)+23/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +23/2\cdot(q,2)\cdot(q-3,5)-98\cdot(q,2)\cdot(q-1,7)+123/2\cdot(q,3)\cdot(q-1,4)\\
&\quad +4983\cdot(q-1,3)\cdot(q-1,4)-1/2\cdot(q-1,3)\cdot(q,5)+2\cdot(q-1,3)\cdot(q-1,5)\\
&\quad -1/2\cdot(q-1,3)\cdot(q-2,5)-1/2\cdot(q-1,3)\cdot(q-3,5)-4493582601/8\cdot(q,2)\\
&\quad -6318\cdot(q,3)+416030747/2\cdot(q-1,3)+456330885/4\cdot(q-1,4)-2387/4\cdot(q,5)\\
&\quad +16882226\cdot(q-1,5)-719\cdot(q-2,5)-719\cdot(q-3,5)-158\cdot(q,7)+4285778/3\cdot(q-1,7)\\
&\quad -445/3\cdot(q-2,7)-158\cdot(q-3,7)-445/3\cdot(q-4,7)-158\cdot(q-5,7)\\
&\quad +24764641/16\cdot(q-1,8)+95/2\cdot(q-3,8)-55415/16\cdot(q-5,8)+171280/3\cdot(q-1,9)\\
&\quad -65/2\cdot(q-2,9)-325/6\cdot(q-4,9)-65/2\cdot(q-5,9)-325/6\cdot(q-7,9)+1678\cdot(q-1,11)\\
&\quad +10\cdot(q-1,13)+9\cdot(q-1,16)+12688773577/8]\cdot q^{10}\\
&+[-29059613\cdot(q,2)\cdot(q-1,3)-181211\cdot(q,2)\cdot(q-1,5)+13/2\cdot(q,2)\cdot(q-2,5)\\
&\quad +13/2\cdot(q,2)\cdot(q-3,5)-3790/3\cdot(q,2)\cdot(q-1,7)-1/3\cdot(q,2)\cdot(q-2,7)\\
&\quad -1/3\cdot(q,2)\cdot(q-4,7)+379/2\cdot(q,3)\cdot(q-1,4)+27533\cdot(q-1,3)\cdot(q-1,4)\\
&\quad -5/2\cdot(q-1,3)\cdot(q,5)+45\cdot(q-1,3)\cdot(q-1,5)-5/2\cdot(q-1,3)\cdot(q-2,5)\\
&\quad -5/2\cdot(q-1,3)\cdot(q-3,5)-530815099\cdot(q,2)-22505/2\cdot(q,3)\\
&\quad +480092685/2\cdot(q-1,3)+264157843/2\cdot(q-1,4)-1597/4\cdot(q,5)+22942200\cdot(q-1,5)\\
&\quad -510\cdot(q-2,5)-510\cdot(q-3,5)-133\cdot(q,7)+2561764\cdot(q-1,7)-117\cdot(q-2,7)\\
&\quad -133\cdot(q-3,7)-117\cdot(q-4,7)-133\cdot(q-5,7)+2912347\cdot(q-1,8)+33\cdot(q-3,8)
\end{aligned}
$$

$$
\begin{aligned}
&-5497 \cdot (q-5,8) + 190171 \cdot (q-1,9) - 63/2 \cdot (q-2,9) - 105/2 \cdot (q-4,9) - 63/2 \cdot (q-5,9)\\
&-105/2 \cdot (q-7,9) + 9551 \cdot (q-1,11) + 190 \cdot (q-1,13) + 174 \cdot (q-1,16) + 5789287499/4] \cdot q^9\\
+ [&-40459553 \cdot (q,2) \cdot (q-1,3) - 549900 \cdot (q,2) \cdot (q-1,5) + 15 \cdot (q,2) \cdot (q-2,5)\\
&+15 \cdot (q,2) \cdot (q-3,5) - 9174 \cdot (q,2) \cdot (q-1,7) + 767/2 \cdot (q,3) \cdot (q-1,4)\\
&-1/2 \cdot (q,3) \cdot (q-2,5) - 1/2 \cdot (q,3) \cdot (q-3,5) + 113088 \cdot (q-1,3) \cdot (q-1,4)\\
&-13/2 \cdot (q-1,3) \cdot (q,5) + 406 \cdot (q-1,3) \cdot (q-1,5) - 7 \cdot (q-1,3) \cdot (q-2,5)\\
&-7 \cdot (q-1,3) \cdot (q-3,5) - 1924232911/4 \cdot (q,2) - 37237/2 \cdot (q,3)\\
&+538946029/2 \cdot (q-1,3) + 146099601 \cdot (q-1,4) - 2855/4 \cdot (q,5) + 29758264 \cdot (q-1,5)\\
&-1611/2 \cdot (q-2,5) - 1611/2 \cdot (q-3,5) - 234 \cdot (q,7) + 12688895/3 \cdot (q-1,7)\\
&-667/3 \cdot (q-2,7) - 234 \cdot (q-3,7) - 667/3 \cdot (q-4,7) - 234 \cdot (q-5,7)\\
&+40503455/8 \cdot (q-1,8) + 188 \cdot (q-3,8) - 56465/8 \cdot (q-5,8) + 509282 \cdot (q-1,9)\\
&-141/2 \cdot (q-2,9) - 235/2 \cdot (q-4,9) - 141/2 \cdot (q-5,9) - 235/2 \cdot (q-7,9)\\
&+38442 \cdot (q-1,11) + 1650 \cdot (q-1,13) + 1602 \cdot (q-1,16) + 2530987067/2] \cdot q^8\\
+ [&-52231527 \cdot (q,2) \cdot (q-1,3) - 1336999 \cdot (q,2) \cdot (q-1,5) + 4 \cdot (q,2) \cdot (q-2,5)\\
&+4 \cdot (q,2) \cdot (q-3,5) - 42806 \cdot (q,2) \cdot (q-1,7) + 1233/2 \cdot (q,3) \cdot (q-1,4)\\
&+354100 \cdot (q-1,3) \cdot (q-1,4) - 16 \cdot (q-1,3) \cdot (q,5) + 2104 \cdot (q-1,3) \cdot (q-1,5)\\
&-16 \cdot (q-1,3) \cdot (q-2,5) - 16 \cdot (q-1,3) \cdot (q-3,5) - 3306752015/8 \cdot (q,2)\\
&-24090 \cdot (q,3) + 290556642 \cdot (q-1,3) + 610810751/4 \cdot (q-1,4) - 446 \cdot (q,5)\\
&+36688541 \cdot (q-1,5) - 495 \cdot (q-2,5) - 495 \cdot (q-3,5) - 174 \cdot (q,7) + 19125631/3 \cdot (q-1,7)\\
&-470/3 \cdot (q-2,7) - 174 \cdot (q-3,7) - 470/3 \cdot (q-4,7) - 174 \cdot (q-5,7)\\
&+128427871/16 \cdot (q-1,8) + 144 \cdot (q-3,8) - 111537/16 \cdot (q-5,8) + 3331663/3 \cdot (q-1,9)\\
&-125/2 \cdot (q-2,9) - 625/6 \cdot (q-4,9) - 125/2 \cdot (q-5,9) - 625/6 \cdot (q-7,9)\\
&+114993 \cdot (q-1,11) + 8480 \cdot (q-1,13) + 8892 \cdot (q-1,16) + 8385535229/8] \cdot q^7\\
+ [&-61465576 \cdot (q,2) \cdot (q-1,3) - 2606987 \cdot (q,2) \cdot (q-1,5) + 3 \cdot (q,2) \cdot (q-2,5)\\
&+3 \cdot (q,2) \cdot (q-3,5) - 407927/3 \cdot (q,2) \cdot (q-1,7) - 5/3 \cdot (q,2) \cdot (q-2,7)\\
&-5/3 \cdot (q,2) \cdot (q-4,7) + 700 \cdot (q,3) \cdot (q-1,4) + 849093 \cdot (q-1,3) \cdot (q-1,4)\\
&-30 \cdot (q-1,3) \cdot (q,5) + 7040 \cdot (q-1,3) \cdot (q-1,5) - 30 \cdot (q-1,3) \cdot (q-2,5)\\
&-30 \cdot (q-1,3) \cdot (q-3,5) - 1326809247/4 \cdot (q,2) - 60189/2 \cdot (q,3)\\
&+294846038 \cdot (q-1,3) + 296566455/2 \cdot (q-1,4) - 2557/4 \cdot (q,5) + 42306362 \cdot (q-1,5)\\
&-1309/2 \cdot (q-2,5) - 1309/2 \cdot (q-3,5) - 253 \cdot (q,7) + 25867609/3 \cdot (q-1,7)\\
&-668/3 \cdot (q-2,7) - 253 \cdot (q-3,7) - 668/3 \cdot (q-4,7) - 253 \cdot (q-5,7)\\
&+90750135/8 \cdot (q-1,8) + 827/2 \cdot (q-3,8) - 34541/8 \cdot (q-5,8) + 5898215/3 \cdot (q-1,9)\\
&-193/2 \cdot (q-2,9) - 965/6 \cdot (q-4,9) - 193/2 \cdot (q-5,9) - 965/6 \cdot (q-7,9)\\
&+260166 \cdot (q-1,11) + 28280 \cdot (q-1,13) + 32424 \cdot (q-1,16) + 811027539] \cdot q^6\\
+ [&-64089333 \cdot (q,2) \cdot (q-1,3) - 4022862 \cdot (q,2) \cdot (q-1,5) - 9/2 \cdot (q,2) \cdot (q-2,5)\\
&-9/2 \cdot (q,2) \cdot (q-3,5) - 300784 \cdot (q,2) \cdot (q-1,7) + 474 \cdot (q,3) \cdot (q-1,4)\\
&+1545061 \cdot (q-1,3) \cdot (q-1,4) - 77/2 \cdot (q-1,3) \cdot (q,5) + 15839 \cdot (q-1,3) \cdot (q-1,5)\\
&-77/2 \cdot (q-1,3) \cdot (q-2,5) - 77/2 \cdot (q-1,3) \cdot (q-3,5) - 1950156283/8 \cdot (q,2)\\
&-65579/2 \cdot (q,3) + 545822287/2 \cdot (q-1,3) + 521609945/4 \cdot (q-1,4) - 1297/4 \cdot (q,5)\\
&+44136546 \cdot (q-1,5) - 635/2 \cdot (q-2,5) - 635/2 \cdot (q-3,5) - 151 \cdot (q,7)\\
&+10153503 \cdot (q-1,7) - 138 \cdot (q-2,7) - 151 \cdot (q-3,7) - 138 \cdot (q-4,7) - 151 \cdot (q-5,7)\\
&+221230211/16 \cdot (q-1,8) + 262 \cdot (q-3,8) + 15619/16 \cdot (q-5,8) + 8339251/3 \cdot (q-1,9)\\
&-137/2 \cdot (q-2,9) - 685/6 \cdot (q-4,9) - 137/2 \cdot (q-5,9) - 685/6 \cdot (q-7,9)\\
&+443263 \cdot (q-1,11) + 63510 \cdot (q-1,13) + 80388 \cdot (q-1,16) + 4613335551/8] \cdot q^5\\
+ [&-56768228 \cdot (q,2) \cdot (q-1,3) - 4777963 \cdot (q,2) \cdot (q-1,5) - 29/2 \cdot (q,2) \cdot (q-2,5)\\
&-29/2 \cdot (q,2) \cdot (q-3,5) - 463157 \cdot (q,2) \cdot (q-1,7) + 42 \cdot (q,3) \cdot (q-1,4)\\
&-3/2 \cdot (q,3) \cdot (q-2,5) - 3/2 \cdot (q,3) \cdot (q-3,5) + 2086551 \cdot (q-1,3) \cdot (q-1,4)\\
&-63 \cdot (q-1,3) \cdot (q,5) + 24132 \cdot (q-1,3) \cdot (q-1,5) - 129/2 \cdot (q-1,3) \cdot (q-2,5)\\
&-129/2 \cdot (q-1,3) \cdot (q-3,5) - 1278466669/8 \cdot (q,2) - 32641 \cdot (q,3)\\
&+220224154 \cdot (q-1,3) + 400048563/4 \cdot (q-1,4) - 334 \cdot (q,5) + 39549073 \cdot (q-1,5)\\
&-627/2 \cdot (q-2,5) - 627/2 \cdot (q-3,5) - 195 \cdot (q,7) + 29802046/3 \cdot (q-1,7)\\
&-530/3 \cdot (q-2,7) - 195 \cdot (q-3,7) - 530/3 \cdot (q-4,7) - 195 \cdot (q-5,7)
\end{aligned}
$$

$$
\begin{aligned}
&\quad +222499445/16 \cdot (q-1,8) + 1083/2 \cdot (q-3,8) + 128685/16 \cdot (q-5,8) + 9132505/3 \cdot (q-1,9) \\
&\quad -88 \cdot (q-2,9) - 440/3 \cdot (q-4,9) - 88 \cdot (q-5,9) - 440/3 \cdot (q-7,9) + 556646 \cdot (q-1,11) \\
&\quad +96780 \cdot (q-1,13) + 136269 \cdot (q-1,16) + 2954375151/8] \cdot q^4 \\
&+ [-40275348 \cdot (q,2) \cdot (q-1,3) - 4176859 \cdot (q,2) \cdot (q-1,5) - 7 \cdot (q,2) \cdot (q-2,5) \\
&\quad -7 \cdot (q,2) \cdot (q-3,5) - 485760 \cdot (q,2) \cdot (q-1,7) - 3 \cdot (q,2) \cdot (q-2,7) \\
&\quad -3 \cdot (q,2) \cdot (q-4,7) - 767/2 \cdot (q,3) \cdot (q-1,4) + 2012401 \cdot (q-1,3) \cdot (q-1,4) \\
&\quad -39 \cdot (q-1,3) \cdot (q,5) + 24576 \cdot (q-1,3) \cdot (q-1,5) - 39 \cdot (q-1,3) \cdot (q-2,5) \\
&\quad -39 \cdot (q-1,3) \cdot (q-3,5) - 719409981/8 \cdot (q,2) - 53495/2 \cdot (q,3) + 145537776 \cdot (q-1,3) \\
&\quad +253319269/4 \cdot (q-1,4) - 245/2 \cdot (q,5) + 28379122 \cdot (q-1,5) - 115 \cdot (q-2,5) \\
&\quad -115 \cdot (q-3,5) - 76 \cdot (q,7) + 22706758/3 \cdot (q-1,7) - 122/3 \cdot (q-2,7) - 76 \cdot (q-3,7) \\
&\quad -122/3 \cdot (q-4,7) - 76 \cdot (q-5,7) + 174158741/16 \cdot (q-1,8) + 485/2 \cdot (q-3,8) \\
&\quad +240957/16 \cdot (q-5,8) + 2463448 \cdot (q-1,9) - 39 \cdot (q-2,9) - 65 \cdot (q-4,9) - 39 \cdot (q-5,9) \\
&\quad -65 \cdot (q-7,9) + 495692 \cdot (q-1,11) + 98460 \cdot (q-1,13) + 155052 \cdot (q-1,16) + 1651610663/8] \cdot q^3 \\
&+ [-21045267 \cdot (q,2) \cdot (q-1,3) - 2505403 \cdot (q,2) \cdot (q-1,5) - 15 \cdot (q,2) \cdot (q-2,5) \\
&\quad -15 \cdot (q,2) \cdot (q-3,5) - 329832 \cdot (q,2) \cdot (q-1,7) - 921/2 \cdot (q,3) \cdot (q-1,4) \\
&\quad +1299982 \cdot (q-1,3) \cdot (q-1,4) - 119/2 \cdot (q-1,3) \cdot (q,5) + 15908 \cdot (q-1,3) \cdot (q-1,5) \\
&\quad -119/2 \cdot (q-1,3) \cdot (q-2,5) - 119/2 \cdot (q-1,3) \cdot (q-3,5) - 81405273/2 \cdot (q,2) \\
&\quad -37421/2 \cdot (q,3) + 144282719/2 \cdot (q-1,3) + 61005621/2 \cdot (q-1,4) - 89 \cdot (q,5) \\
&\quad +14860042 \cdot (q-1,5) - 147/2 \cdot (q-2,5) - 147/2 \cdot (q-3,5) - 197/2 \cdot (q,7) \\
&\quad +12335137/3 \cdot (q-1,7) - 553/6 \cdot (q-2,7) - 197/2 \cdot (q-3,7) - 553/6 \cdot (q-4,7) \\
&\quad -197/2 \cdot (q-5,7) + 24458123/4 \cdot (q-1,8) + 391 \cdot (q-3,8) + 82619/4 \cdot (q-5,8) \\
&\quad +4110983/3 \cdot (q-1,9) - 50 \cdot (q-2,9) - 250/3 \cdot (q-4,9) - 50 \cdot (q-5,9) - 250/3 \cdot (q-7,9) \\
&\quad +293768 \cdot (q-1,11) + 63870 \cdot (q-1,13) + 112797 \cdot (q-1,16) + 189458755/2] \cdot q^2 \\
&+ [-7072020 \cdot (q,2) \cdot (q-1,3) - 911691 \cdot (q,2) \cdot (q-1,5) - 5/2 \cdot (q,2) \cdot (q-2,5) \\
&\quad -5/2 \cdot (q,2) \cdot (q-3,5) - 130273 \cdot (q,2) \cdot (q-1,7) - 569/2 \cdot (q,3) \cdot (q-1,4) \\
&\quad +499822 \cdot (q-1,3) \cdot (q-1,4) - 14 \cdot (q-1,3) \cdot (q,5) + 5946 \cdot (q-1,3) \cdot (q-1,5) \\
&\quad -14 \cdot (q-1,3) \cdot (q-2,5) - 14 \cdot (q-1,3) \cdot (q-3,5) - 52374147/4 \cdot (q,2) - 8587 \cdot (q,3) \\
&\quad +23329394 \cdot (q-1,3) + 9718348 \cdot (q-1,4) - 19 \cdot (q,5) + 4931122 \cdot (q-1,5) - 15 \cdot (q-2,5) \\
&\quad -15 \cdot (q-3,5) - 33/2 \cdot (q,7) + 1389371 \cdot (q-1,7) - 19/2 \cdot (q-2,7) - 33/2 \cdot (q-3,7) \\
&\quad -19/2 \cdot (q-4,7) - 33/2 \cdot (q-5,7) + 17346667/8 \cdot (q-1,8) + 179/2 \cdot (q-3,8) \\
&\quad +191607/8 \cdot (q-5,8) + 462780 \cdot (q-1,9) - 9 \cdot (q-2,9) - 15 \cdot (q-4,9) - 9 \cdot (q-5,9) \\
&\quad -15 \cdot (q-7,9) + 103194 \cdot (q-1,11) + 23840 \cdot (q-1,13) + 47214 \cdot (q-1,16) + 125480723/4] \cdot q \\
&+ [-1129282 \cdot (q,2) \cdot (q-1,3) - 150412 \cdot (q,2) \cdot (q-1,5) - 4 \cdot (q,2) \cdot (q-2,5) \\
&\quad -4 \cdot (q,2) \cdot (q-3,5) - 67958/3 \cdot (q,2) \cdot (q-1,7) - 5/3 \cdot (q,2) \cdot (q-2,7) \\
&\quad -5/3 \cdot (q,2) \cdot (q-4,7) - 98 \cdot (q,3) \cdot (q-1,4) - (q,3) \cdot (q-2,5) - (q,3) \cdot (q-3,5) \\
&\quad +85809 \cdot (q-1,3) \cdot (q-1,4) - 41/2 \cdot (q-1,3) \cdot (q,5) + 952 \cdot (q-1,3) \cdot (q-1,5) \\
&\quad -43/2 \cdot (q-1,3) \cdot (q-2,5) - 43/2 \cdot (q-1,3) \cdot (q-3,5) - 2252960 \cdot (q,2) - 3319 \cdot (q,3) \\
&\quad +7251607/2 \cdot (q-1,3) + 1443592 \cdot (q-1,4) - 9 \cdot (q,5) + 765104 \cdot (q-1,5) - 5/2 \cdot (q-2,5) \\
&\quad -5/2 \cdot (q-3,5) - 24 \cdot (q,7) + 649076/3 \cdot (q-1,7) - 22/3 \cdot (q-2,7) - 24 \cdot (q-3,7) \\
&\quad -22/3 \cdot (q-4,7) - 24 \cdot (q-5,7) + 376547 \cdot (q-1,8) + 118 \cdot (q-3,8) + 25465 \cdot (q-5,8) \\
&\quad +212593/3 \cdot (q-1,9) - 13 \cdot (q-2,9) - 65/3 \cdot (q-4,9) - 13 \cdot (q-5,9) - 65/3 \cdot (q-7,9) \\
&\quad +16170 \cdot (q-1,11) + 3890 \cdot (q-1,13) + 8619 \cdot (q-1,16) + 5606608]
\end{aligned}
$$

Chapter B

Codes and GAP functions

We append in this appendix the most important functions of our algorithm which are summarised in pseudo-code. We use the same name of functions as in our GAP implementation. The functions and variables are typeset in typewriter as `NumberOfClassTwoAlgebras( r, action )`.

Algorithm 2: `KFunctions( S, m )`
Computation of the characteristic function of a residue class
(Note: In GAP the set `S` of prime divisors of `m` is given as input.)

```
Input: natural number m;
Output: functions k_{l,m}(x) for all 1 ≤ l ≤ m;

   ## Check if these functions have been computed before.              ##
if IsBound(KLIST[m]) then return KLIST[m]; fi;

   ## Otherwise compute the functions (functions will be stored in a list k)   ##
k := an empty list of length m;
S := set of all prime divisors of m;
for 1 ≤ i ≤ m do
   k[i] := Σ_{T⊆S} (-1)^|T| gcd(x - i, m/d_T);
      ## Normalise by dividing the k'th function by its value for x=i   ##
   k[i] := k[i]/k[i][i];
od;

   ## Store and return                                                 ##
KLIST[m] := k;
return k;
```

Algorithm 3: `GcdOnPoints( list )`
Computation of the point-wise greatest common divisor of a list of rational polynomials.

```
Input: f_1, ..., f_s ∈ ℚ[x];
Output: pgcd(f_1(q), ..., f_s(q));

   ## Use Euclidean algorithm:                                          ##
f := greatest common divisor of all f_i over ℚ[x];      # choose f primitive
Compute polynomials g_i ∈ ℚ[x] such that Σ_{i=1}^{s} g_i·f_i = f;

   ## Compute modulus:                                                  ##
m := least common multiple of all denominators of coefficients of all g_i;

   ## Determine PORC part:                                              ##
f_i := f_i/f;
k := list of all characteristic functions for residue classes modulo m;   # see algorithm 2
d := empty list of length m;
for 1 ≤ i ≤ m do
   x := smallest positive integer with f_i(x) ≠ 0 and x ≡ i mod m;
   d[i] := ggT( f_1(i), ..., f_s(i) );
od;
k := Σ_{i=1}^{s} k[i] · d[i];
return f · k;
```

Algorithm 4: `GetJNFTypes( n )`
Determination of all JNF-types for a fixed natural number **n**.

```
Input: natural number n;
Output: set of all types of GL(n, q);
T := ∅;                                                         # Store JNF-types in T
M := set of all partitions of numbers j with 1 ≤ j ≤ n;
for P = (p_1, ... , p_k) ∈ M do      # loop over all partitions P and consider them as k-tuple
    A(P) := set of all a ∈ ℕ^k with  Σ_{i=1}^{k} a_ip_i = n;
    for a = (a_1, ... , a_k) ∈ A(P) do
        C := X_{i=1}^{k} B_i where B_i is the set of all partitions of a_i;
        for c = (c_1, ..., c_k) ∈ C do
            t := ((c_1, p_1), ..., (c_k, p_k));
            Append t to T;
        od;
    od;
od;
Reduce T by symmetry;                                           # Prepare output
for t in T do
    Make t to record and append additional information .clen and .ss;
od;
return T;
```

Algorithm 5: `ExceptionalPrimesAssociative( n )`
Computation of all exceptional primes such that Theorem 5.6 cannot be applied to matrices over fields with those characteristics.

```
function FindSequenceNumber(alpha, beta, n)
        ## Initialise the function FindSequenceNumberByRecursion              ##
        #  No sequences have been found so far, hence it is count = 0
        #  We start to find first possible sequence, hence it is depth = 1
    return FindSequenceNumberByRecursion(alpha, beta, 0, 1, n);
end

function ExceptionalPrimesAssociative(n)
        ## Initialise the function FindSequenceNumber                         ##
        #  The first sequence alpha and the last sequence beta are generated as in the proofs
        #  Calls the function FindSequenceNumber for all natural numbers up to n
    m := ∅;
    for 1 ≤ i ≤ n do
        c := FindSequenceNumber({1,...,i}, {n-i+1,...,n}, n);
        Add all prime divisors of c to m;
    od;
    return All primes less than or equal to the maximum in m;
end
```

Algorithm 6: FindSequenceNumberByRecursion(alpha, beta, count, depth, max)
Some helper function to compute the exceptional prime numbers.

```
function FindSequenceNumberByRecursion(alpha, beta, count, depth, max)
    l    := Number of elements in alpha;
    iter := Set {0,1}^l;
    if depth < max then
        ## As long as maximal depth is not reached add (if possible) elements from iter to
           alpha ##
        for add in iter do
            if IsValidNextStep(alpha, add, beta, depth, max) then
                ## Repeat as long as maximal depth is not reached ##
                count := FindSequenceNumberByRecursion(alpha+add, beta, count,
                 depth+1, max);
            fi;
        od;
    else
        for add in iter do
            ## Increase counter by 1 if the step at the maximal depth is allowed ##
            if IsValidLastStep(alpha, add, beta) then
                count := count +1;
            fi;
        od;
    fi;
    return count;
end
```

Algorithm 7: `DetermineEqns( type, act )`
Computation of all types of matrices in $\mathcal{O}$(`type`) under the action `act`

Input: type `t`, action `act`;
Output: All possible types $\bar{t}$ of elements in $\mathcal{O}(t)$ and their number of occurences $\mathcal{A}(t,\bar{t})$;

```
  ## Get eigenvalues depending on given action and add information to the type   ##
AddEigenvalues(type, act);

  ## Setup and preparation                                                        ##
fdes := Equations describing the fields;
ineq := Equations describing the restrictions;
pows := The power set of ineq togethter with its incidence matrix;
cm   := List of all possible equations between the eigenvalues and a group of possible
  permutations;
l    := Number of equations in cm;

  ## Initialisation for loop                                                      ##
os[1] := {∅};                                                    # Store the orbits
rs[1] := {ElmNumbers( [], fdes, ineq, pows )};                   # Store the numbers
av    := {∅};                          # Store combinations yielding no solutions

  ## Loop over size of sets of equations                                          ##
for 1 ≤ i ≤ l do
  | ps := CombinationsAvoid(l, av, ps, i)        # Get possible combinations of i equations
  | if |ps| = 0 then break; fi;        # leave the loop if no possible equation is left over
  | os[i+1] := Orbits(cm.symm, ps, OnSets);
  | for 1 ≤ j ≤ Length(os[i+1]) do                     # loop over the computed orbits
  | | tr := Get all equations from cm decoded by a representative of the considered orbit;
  | | rs[i+1][j] := ElmNumbers(tr, fdes, ineq, pows);
  | od;
  | if  rs[i+1][j] = 0 then  Append(av, os[i+1][j]) fi ;   # check if orbits can be avoided
od;

  ## Inclusion-exclusion principle                                                ##
  #  rs[i][j] now stores the numbers |K_I(U)|.
  #  Apply inclusion-exclusion to obtain A(t, t̄(U)), U corresponds to the sets stored in os[i][j].
rs[i][j] ← Application of inclusion-exclusion;

  ## Prepare output                                                               ##
Remove all trivial cases;
Combine the numbers in rs[i][j] and the equation in os[i][j] to a record;
result := List of all records;

return  result;
```

Bibliography

[1] G. Bagnera. *La composizione dei gruppi finiti il cui grado e la quinta potenza di un numero primo*. Bernardoni di C, rebeschini, 1898.

[2] G. Bagnera. Sopra i gruppi astratti di grado 32. *Annali di Mat. (3)*, 2:263–275, 1899.

[3] H. U. Besche, B. Eick, and E. A. O'Brien. A millennium project: Constructing small groups. *IJAC*, 12(5):623–644, 2002.

[4] N. Blackburn. On a special class of p-groups. *Acta Math.*, 100:45–92, 1958.

[5] W. Burnside. *Theory of groups of finite order*. Dover Publications, Inc., New York, 1955. 2d ed.

[6] W. A. De Graaf. Classification of nilpotent associative algebras of small dimension. *Internat. J. Algebra Comput.*, 28(1):133–161, 2018.

[7] M. du Sautoy and M. Vaughan-Lee. Non-PORC behaviour of a class of descendant p-groups. *J. Algebra*, 361:287–312, 2012.

[8] D. S. Dummit and R. M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[9] B. Eick. Computing automorphism groups and testing isomorphisms for modular group algebras. *J. Algebra*, 320(11):3895–3910, 2008.

[10] B. Eick and E. A. O'Brien. Enumerating p-groups. *J. Austral. Math. Soc. Ser. A*, 67(2):191–205, 1999. Group theory.

[11] B. Eick and E. A. O'Brien. *AutPGrp - Computing the automorphism group of a p-group*. The GAP Group, 2005. A refereed GAP 4 package.

[12] B. Eick and M. Wesche. *ClassTwoAlg - Computing PORC polynomials for nilpotent associative algebras of class* 2 *over finite fields*. The GAP Group, 2017. A GAP 4 package.

[13] B. Eick and M. Wesche. Enumeration of nilpotent associative algebras of class 2 over arbitrary finite fields. *J. Algebra*, 503:573–589, 2018.

[14] G. Fischer and R. Sacher. *Einführung in die Algebra*. B. G. Teubner, Stuttgart, 1978. Zweite überarbeitete Auflage, Teubner Studienbücher: Mathematik.

[15] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.8.6*, 2016.

[16] J. A. Green. The characters of the finite general linear groups. *Trans. Amer. Math. Soc.*, 80:402–447, 1955.

[17] G. Higman. Enumerating p-groups. I. Inequalities. *Proc. London Math. Soc. (3)*, 10:24–30, 1960.

[18] G. Higman. Enumerating p-groups. II. Problems whose solution is PORC. *Proc. London Math. Soc. (3)*, 10:566–582, 1960.

[19] N. Jacobson. *Basic algebra. I*. W. H. Freeman and Company, New York, second edition, 1985.

[20] R. James. *The Groups of Order p^6*. PhD thesis, University of Sydney, 1968.

[21] R. James. The groups of order p^6 (p an odd prime). *Math. Comp.*, 34(150):613–637, 1980.

[22] R. James, M. Newman, and E. A. O'Brien. The groups of order 128. *Journal of Algebra*, 129(1):136 – 158, 1990.

[23] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[24] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, second edition, 2015. With contribution by A. V. Zelevinsky and a foreword by Richard Stanley, Reprint of the 2008 paperback edition.

[25] M. Marcus. *Finite dimensional multilinear algebra. Part I*. Marcel Dekker, Inc., New York, 1973. Pure and Applied Mathematics, Vol. 23.

[26] M. Marcus. *Finite dimensional multilinear algebra. Part II*. Marcel Dekker, Inc., New York, 1975. Pure and Applied Mathematics, Vol. 23.

[27] M. F. Newman, E. A. O'Brien, and M. Vaughan-Lee. Groups and nilpotent Lie rings whose order is the sixth power of a prime. *J. Algebra*, 278(1):383–401, 2004.

[28] E. A. O'Brien. The p-group generation algorithm. *J. Symbolic Comput.*, 9(5-6):677–698, 1990. Computational group theory, Part 1.

[29] E. A. O'Brien and M. Vaughan-Lee. The groups with order p^7 for odd prime p. *J. Algebra*, 292(1):243–258, 2005.

[30] R. S. Pierce. *Associative algebras*, volume 88 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982. Studies in the History of Modern Science, 9.

[31] J. J. Rotman. *An introduction to the theory of groups*, volume 148 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, fourth edition, 1995.

[32] O. Schreier. Über die Erweiterung von Gruppen. II. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 4(1):321–346, dec 1925.

[33] N. J. A. Sloane and M. Bernstein. The On-Line Encyclopedia of Integer Sequences. `https://oeis.org/A003606`, 2017. The number of types of conjugacy classes in GL(n,q) (this is independent of q).

[34] M. Vaughan-Lee. Choosing elements from finite fields. November 2012.

[35] M. Vaughan-Lee. Graham Higman's PORC conjecture. *Jahresber. Dtsch. Math.-Ver.*, 114(2):89–106, 2012.

[36] M. Vaughan-Lee. On Graham Higman's famous PORC paper. *Int. J. Group Theory*, 1(4):65–79, 2012.

[37] M. Vaughan-Lee. Enumerating algebras over a finite field. *Int. J. Group Theory*, 2(3):49–61, 2013.

[38] B. E. Witty. *Enumeration of groups of prime-power order*. PhD thesis, Australian National University, August 2006.

Nomenclature

(,) Abbreviation for the greatest common divisor, $(x, y) = \gcd(x, y)$, page 3

$\mathcal{A}^*$ Covering algebra of an algebra $\mathcal{A}$, page 25

$A(t, \bar{t})$ Number of representative for conjugacy classes of matrices of type $\bar{t}$ which have type $\bar{t}$ in $\mathcal{O}(t)$, page 29

$\mathrm{CAut}(\mathcal{A}, \mathcal{B})$ Group of central automorphism of $\mathcal{A}$ in $\mathcal{B}$, that is for an $\mathbb{F}_q$-algebra $\mathcal{A}$ and a zero-ideal $\mathcal{B} \leq \mathcal{A}^2$ defined as $\left\{\alpha \in \mathrm{Aut}(A) \mid \alpha|_{\mathcal{A}/\mathcal{B}} = \mathrm{id}|_{\mathcal{A}/\mathcal{B}}\right\}$, page 26

$\mathcal{C}_{a(x)}$ Companion matrix of the polynomial $a(x)$, page 16

$C_G(a)$ Centraliser of a group element $a \in G$ within the group G, $C_G(a) := \{g \in G \mid gag^{-1} = a\}$, page 18

c_t Order of the centraliser of a matrix of type t, page 18

$D(t)$ Set of eigenvalues of a matrix $g \otimes g$ in Jordan normal form where g is a matrix of type t, page 38

$\mathcal{E}(t)$ The set of all matrices in $\mathrm{GL}(n, q)$ of type t, page 18

$J_r(a)$ Jordan block of size r with eigenvalue a, page 33

$K_I(U)$ Set of elements satisfying all equations in U, but none in I, page 41

$\breve{K}_J(U)$ Set of elements satisfying all equations in U and in J, page 41

$k_{l,m}(x)$ Characteristic function of the residue class of l modulo m, page 5

λ A partition $\lambda = (\lambda_1, \lambda_2, \ldots,)$ with $\lambda_{i+1} \geq \lambda_i \geq 0$ for all i and $\lambda_i \in \mathbb{N} \cup \{0\}$, page 13

λ' The partition conjugate to λ, page 13

$L(t)$ Set of all possible equations amongst the elements of $D(t)$, page 39

$\mathcal{M}$ Multiplicator of an algebra $\mathcal{A}$, page 25

$m_i(\lambda)$ Multiplicity of part i in λ; that is how often i appears in λ, page 13

$\breve{M}_J(U)$	Matrix describing all elements determined by the set $\breve{K}_J(U)$, page 42
μ_i	Given a natural number n_i, the set μ_i contains all maximal, proper divisor of n_i, page 40
$\mathcal{N}$	Nucleus of an algebra $\mathcal{A}$, page 26
$N_{d,r}(q)$	Number of isomorphism types of algebras of class 2, dimension d, and rank r, page 1
$n(\lambda)$	Function n defined by $n(\lambda) = \sum(i-1)\lambda_i$, page 14
$\mathcal{O}(t)$	Set of all matrices of type t after the application of the Kronecker product, $\{g \otimes g \mid g \in \mathcal{E}(t)\}$, page 29
pgcd	point-wise greatest common divisor, page 7
$\pi(t)$	Number of permutations of the type-parameters of t that do not change the type t, page 20
t	Type of a matrix, page 18
$T(r,q)$	The tensor product space $\mathbb{F}_q^r \otimes_{\mathbb{F}_q} \mathbb{F}_q^r$, page 27
$\mathcal{U}_k(r,q)$	Set of all k-dimensional subspaces of the vector space $T(r,q)$, page 27

Index